An Introduction to Locally Convex Cones

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Abstract

This survey introduces and motivates the foundations of the theory of locally convex cones which aims to generalize the well-established theory of locally convex topological vector spaces. We explain the main concepts, provide definitions, principal results, examples and applications. For details and proofs we generally refer to the literature.

Index Terms: cone-valued functions, locally convex cones, Korovkin type approximation

1. Introduction

Endowed with suitable topologies, vector spaces yield rich and well-considered structures. Locally convex topological vector spaces in particular permit an effective duality theory whose study provides valuable insight into the spaces themselves. Some important mathematical settings, however – while close to the structure of vector spaces – do not allow subtraction of their elements or multiplication by negative scalars. Examples are certain classes of functions that may take infinite values or are characterized through inequalities rather than equalities. They arise naturally in integration and in potential theory. Likewise, families of convex subsets of vector spaces which are of interest in various contexts do not form vector spaces. If the cancellation law fails, domains of this type may not even be embedded into larger vector spaces in order to apply results and techniques from classical functional analysis. They merit the investigation of a more general structure.

The theory of locally convex cones as developed in [7] admits most of these settings. A topological structure on a cone is introduced using order-theoretical concepts. Staying reasonably close to the theory of locally convex spaces, this approach yields a sufficiently rich duality theory including Hahn-Banach type extension and separation theorems for linear functionals. In this article we shall give an outline of the principal concepts of this emerging theory. We survey the main results including some yet unpublished ones and provide primary examples and applications. However, we shall generally refrain from supplying technical details and proofs but refer to different sources instead.

2. Ordered cones and monotone linear functionals

A *cone* is a set P endowed with an addition

$$(a, b) \to a + b$$

and a scalar multiplication

$$(\alpha, a) \to \alpha a$$

for $a \in P$ and real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in P$, that is:

$$(a + b) + c = a + (b + c) \quad \text{for all } a, b, c \in P$$
$$a + b = b + a \quad \text{for all } a, b \in P$$
$$0 + a = a \quad \text{for all } a \in P$$

For the scalar multiplication the usual associative and distributive properties hold, that is:

$$\alpha(\beta a) = (\alpha\beta)a \quad \text{for all } \alpha, \beta \geq 0 \text{ and } a \in P$$

$(\alpha+\beta)a = \alpha a + \beta a$	for all $\alpha, \beta \geq 0$ and $a \in P$
$\alpha(a+b) = \alpha a + \alpha b$	for all $\alpha \geq 0$ and $a, b \in P$
$1a = a$	for all $a \in P$
$0a = 0$	for all $a \in P$

Unlike the situation for vector spaces, the condition $0a = 0$ needs to be stated independently for cones, as it is not a consequence of the preceding requirements (see [6]). The *cancellation law*, stating that

(C) $\quad a + c = b + c \quad$ implies that $\quad a = b$

however, is not required in general. It holds if and only if the cone P can be embedded into a real vector space.

A *subcone* Q of a cone P is a non-empty subset of P that is closed for addition and multiplication by non-negative scalars.

An *ordered cone* P carries additionally a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is

$a \leq b$ implies that $\; a + c \leq b + c\;$ and $\;\alpha a \leq \alpha b$

for all $a, b, c \in P$ and $\alpha \geq 0$. As equality in P is obviously such an order, all our results about ordered cones will apply to cones without order structures as well. We provide a few examples:

2.1 Examples. (a) In $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$.

(b) Let P be a cone. A subset A of P is called *convex* if

$$\alpha a + (1-\alpha)b \in A$$

whenever $a, b \in A$ and $0 \leq \alpha \leq 1$.We denote by $Conv(P)$ the set of all non-empty convex subsets of P. With the addition and scalar multiplication defined as usual by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

for $A, B \in Conv(P)$, and

$$\alpha A = \{\alpha a \mid a \in A\}$$

for $A \in Conv(P)$ and $\alpha \geq 0$, it is easily verified that $Conv(P)$ is again a cone. Convexity is required to show that $(\alpha+\beta)A$ equals $\alpha A + \beta A$. The set inclusion defines a suitable order on $Conv(P)$ that is compatible with these algebraic operations. The cancellation law generally fails for $Conv(P)$.

(c) Let P be an ordered cone, X any non-empty set. For P-valued functions on X the addition, scalar multiplication and order may be defined pointwise. The set $F(X, P)$ of all such functions again becomes an ordered cone for which the cancellation law holds if and only if it holds for P.

A *linear functional* on a cone P is a mapping $\mu: P \to \overline{\mathbb{R}}$ such that

$\mu(a+b) = \mu(a) + \mu(b) \quad$ and $\quad \mu(\alpha a) = \alpha\mu(a)$

holds for all $a, b \in P$ and $\alpha \geq 0$. Note that linear functionals take only finite values in invertible elements of P. If P is ordered, then μ is called *monotone* if

$a \leq b \quad$ implies that $\quad \mu(a) \leq \mu(b)$.

In various places the literature deals with linear functionals on cones that take values in $\mathbb{R} \cup \{-\infty\}$ (see [6]) instead. In vector spaces both approaches coincide, as linear functionals can take only finite values there, but in applications for cones the value $+\infty$ arises more naturally.

The existence of sufficiently many monotone linear functionals on an ordered cone is guaranteed by a Hahn-Banach type sandwich theorem whose proof may be found in [13] or in a rather weaker version in [7]. It is the basis for the duality theory of ordered cones. In this context, a *sublinear functional* on a cone P is a mapping $p : P \to \overline{\mathbb{R}}$ such that

$p(\alpha a) = \alpha p(a)$ and $p(a + b) \leq p(a) + p(b)$

holds for all $a, b \in P$ and $\alpha \geq 0$. Likewise, a *superlinear functional* on P is a mapping $q : P \to \overline{\mathbb{R}}$ such that

$q(\alpha a) = \alpha q(a)$ and $q(a + b) \geq q(a) + q(b)$ holds for all $a, b \in P$ and $\alpha \geq 0$. Note that superlinear functionals can assume only finite values in invertible elements of P.

It is convenient to use the pointwise order relation for functions f, g on P; that is we shall write $f \leq g$ to abbreviate $f(a) \leq g(a)$ for all $a \in P$.

2.2 Sandwich Theorem (algebraic). *Let P be an ordered cone and let $p : P \to \overline{\mathbb{R}}$ be a sublinear and $q : P \to \overline{\mathbb{R}}$ a superlinear functional such that*

$q(a) \leq p(b)$ *whenever* $a \leq b$ *for* $a, b \in P$.

There exists a monotone linear functional $\mu: P \to \overline{\mathbb{R}}$ such that $q \leq \mu \leq p$.

Note that the above condition for q and p is fulfilled if $q \leq p$ and if one of these functionals is monotone. The superlinear functional q may however not be omitted altogether (or equivalently, replaced by one that also allows the value $-\infty$) without further assumptions. (see Example 2.2 in [13].)

3. Locally convex cones

Because subtraction and multiplication by negative scalars are generally not available, a topological structure for a cone should not be expected to be invariant under translation and scalar multiplication. There are various equivalent approaches to *locally convex cones* as outlined in [7]. The use of *convex quasiuniform structures* is motivated by the following features of neighborhoods in a cone: With every $\overline{\mathbb{R}}$-valued monotone linear functional μ on an ordered cone P we may associate a subset

$$v = \{(a, b) \in P^2 |\, \mu(a) \leq \mu(b) + 1\}$$

of P^2 with the following properties:

(U1) v is convex.
(U2) If $a \leq b$ for $a, b \in P$, then $(a, b) \in v$.
(U3) If $(a, b) \in \lambda v$ and $(b, c) \in \rho v$ for $\lambda, \rho > 0$, then $(a, c) \in (\lambda + \rho)v$.
(U4) For every $b \in P$ there is $\lambda \geq 0$ such that $(0, b) \in \lambda v$.

Any subset v of P^2 with the above properties (U1) to (U4) qualifies as a *uniform neighborhood* for P, and any family V of such neighborhoods fulfilling the usual conditions for a quasiuniform structure, that is:

(U5) For $u, v \in V$ there is $w \in V$ such that $w \subset u \cap v$.
(U6) $\lambda v \in V$ for all $v \in V$ and $\lambda > 0$.

generates a *locally convex cone* (P, V) as elaborated in [7]. More specifically, V creates three hyperspace topologies on P and every $v \in V$ defines neighborhoods for an element $a \in P$ by

$v(a) = \{b \in P |\, (b, a) \in \lambda v \text{ for all } \lambda > 1\}$
in the *upper topology*
$(a)v = \{b \in P |\, (a, b) \in \lambda v \text{ for all } \lambda > 1\}$
in the *lower topology*
$$v(a)v = v(a) \cap (a)v$$
in the *symmetric topology*

However, it is convenient to think of a locally convex cone (P, V) as a subcone of a *full locally convex cone* $\tilde{P}$, i.e. a cone that contains the neighborhoods v as positive elements (see [7], Ch. I).

Referring to the order in $\tilde{P}$, the relation $a \in v(b)$ may be reformulated as $a \leq b + v$. This leads to a second and equivalent approach to locally convex cones that uses the order structure of a larger full cone in order to describe the topology of P (for relations between order and topology we refer to [9]). Let us indicate how this full cone $\tilde{P}$ may be constructed (for details, see [7], Ch. I.5): For a fixed neighborhood $v \in V$ set

$$\tilde{P} = \{a \oplus \alpha v |\, a \in P, 0 \leq \alpha < +\infty\}.$$

We use the obvious algebraic operations on $\tilde{P}$ and the order

$$a \oplus \alpha v \leq b \oplus \beta v$$

if either $\alpha = \beta$ and $a \leq b$, or $\alpha < \beta$ and $(a, b) \in \lambda v$ for all $\lambda > \beta - \alpha$. The embedding $a \to a \oplus 0v$ preserves the algebraic operations and the order of P. The procedure for embedding a locally convex cone (P, V) into a full cone $(\tilde{P}, V)$ that contains a whole system V of neighborhoods as positive elements is similar and elaborated in Ch. I.5 of [7]. The quasiuniform structure of P may then be recovered through the subsets

$$\{(a, b) \in P^2 |\ a \leq b + v\} \subset P^2$$

corresponding to the neighborhoods $v \in V$.

We shall in the following use this order-theoretical approach: We may always assume that a given locally convex cone (P, V) is a subcone of a full locally convex cone $(\tilde{P}, V)$ that contains all neighborhoods as positive elements, and we shall use the order of the latter to describe the topology of P. The above conditions (U1) to (U6) for the quasiuniform structure on P equivalently translate into conditions involving the order relation of $\tilde{P}$ as follows:

(V1) $v \geq 0$ for all $v \in V$.
(V2) For $u, v \in V$ there is $w \in V$ such that $w \leq u$ and $w \leq v$.
(V3) $\lambda v \in V$ whenever $v \in V$ and $\lambda > 0$.
(V4) For $v \in V$ and every $a \in P$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$.

Condition (V4) states that every element $a \in P$ is *bounded below*.

3.1 Examples. (a) The ordered cone $\overline{\mathbb{R}}$ endowed with the neighborhood system $V = \{\varepsilon \in \mathbb{R} |\ \varepsilon > 0\}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} |\ a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $V = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$.

(c) Let $(E, V, \leq)$ be a locally convex ordered topological vector space, where V is a basis of closed, convex, balanced and order convex neighborhoods of the origin in E. Recall that equality is an order relation, hence this example will cover locally convex spaces in general. In order to interpret E as a locally convex cone we shall embed it into a larger full cone. This is done in a canonical way: Let P be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is

$$\alpha A = \{\alpha a |\ a \in A\} \quad \text{and}$$
$$A + B = \{a + b |\ a \in A \text{ and } b \in B\}$$

for $A, B \in P$ and $\alpha \geq 0$. We define the order on P by

$$A \leq B \quad \text{if} \quad A \subset \downarrow B = B + E_-$$

where $E_- = \{x \in E |\ x \leq 0\}$ is the negative cone in E. The requirements for an ordered cone are easily checked. The neighborhood system in P is given by the neighborhood basis $V \subset P$. We observe that for every $A \in P$ and $v \in V$ there is $\rho > 0$ such that $\rho v \cap A \neq \emptyset$. This yields $0 \in A + \rho v$. Therefore $\{0\} \leq A + \rho v$, and every element $A \in P$ is indeed bounded below. Thus (P, V) is a full locally convex cone. Via the embedding $x \to \{x\} : E \to P$ the space E itself is a subcone of P. This embedding preserves the order structure of E, and on its image the symmetric topology of P coincides with the given vector space topology of E. Thus E is indeed a locally convex cone, but not a full cone.

(d) The preceding procedure can be applied to locally convex cones in general. Let (P,V) be a locally convex cone and let $Conv(P)$ denote the cone of all non-empty convex subsets of P, endowed with the canonical order, that is

$$A \leq B \quad \text{if for every} \quad a \in A \quad \text{there is} \quad b \in B \quad \text{such that} \quad a \leq b$$

for $A, B \subset P$. The neighborhood $v \in V$ is defined as a neighborhood for $Conv(P)$ by

$$A \leq B + v \quad \text{if for every } a \in A \text{ there is } b \in B \text{ such that } a \leq b + v$$

The requirements for a locally convex cone are easily checked for $(Conv(P), V)$, and (P,V) is identified with a subcone of $(Conv(P), V)$. Other subcones of $Conv(P)$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $Conv(P)$, respectively. Details on these and further related examples may be found in [7] and [17].

(e) Let (P,V) be a locally convex cone, X a set and let $F(X,P)$ be the cone of all P-valued functions on X, endowed with the pointwise operations and order. If $\bar{P}$ is a full cone containing both P and V then we may identify the elements $v \in V$ with the constant functions $x \to v$ for all $x \in X$, hence V is a subset and a neighborhood system for $F(X,\bar{P})$. A function $f \in F(X,\bar{P})$ is uniformly bounded below, if for every $v \in V$ there is $\rho \geq 0$ such that $0 \leq f + \rho v$. These functions form a full locally convex cone $(F_b(X,\bar{P}), V)$, carrying the topology of uniform convergence. As a subcone, $(F_b(X,\bar{P}), V)$ is a locally convex cone. Alternatively, a more general neighborhood system V_Y for $F(X,P)$ may be created using a suitable family Y of subsets y of X, directed downward with respect to set inclusion, and the neighborhoods v_y for $v \in V$ and $y \in Y$, defined for functions $f, g \in F(X,P)$ as

$$f \leq g + v_y \quad \text{if} \quad f(x) \leq g(x) + v \quad \text{for all } x \in y.$$

In this case we consider the subcone $F_{by}(X,P)$ of all functions in $F(X,P)$ that are uniformly bounded below on the sets in Y. Together with the neighborhood system V_Y, it forms a locally convex cone. $(F_{by}(X,P), V_Y)$ carries the topology of uniform convergence on the sets in Y.

(f) For $x \in \overline{\mathbb{R}}$ denote $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$. For $1 \leq p \leq +\infty$ and a sequence $(x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ let $\|x_i\|_p$ denote the usual l^p norm, that is

$$\|(x_i)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{(1/p)} \in \overline{\mathbb{R}}$$

for $p < +\infty$, and

$$\|(x_i)\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\} \in \overline{\mathbb{R}}.$$

Now let C^p be the cone of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ such that $\|(x_i)\|_p < +\infty$. We use the pointwise order in C^p and the neighborhood system $V_p = \{\rho v_p \mid \rho > 0\}$, where

$$(x_i)_{i\in\mathbb{N}} \leq (y_i)_{i\in\mathbb{N}} + \rho v_p$$

means that $\|(x_i - y_i)^+\|_p \leq \rho$. (In this expression the l^p norm is evaluated only over the indices $i \in \mathbb{N}$ for which $y_i < +\infty$.) It can be easily verified that (C^p, V_p) is a locally convex cone. In fact (C^p, V_p) can be embedded into a full cone following a procedure analogous to that in 2.1 (c). The case for $p = +\infty$ is of course already covered by Part (d).

4. Continuous linear functionals and Hahn-Banach type theorems

A linear functional μ on a locally convex cone (P,V) is said to be *(uniformly) continuous with respect to a neighborhood* $v \in V$ if

$$\mu(a) \leq \mu(b) + 1 \quad \text{whenever} \quad a \leq b + v.$$

Continuity implies that the functional μ is monotone, even with respect to the global preorder $\lesssim$, and takes only finite values in bounded elements $b \in \mathcal{B}$ (see Section 5 below).

The set of all linear functionals μ on P which are continuous with respect to a certain neighborhood v is called the *polar* of v in P and denoted by v_P° (or v° for short). Endowed with the canonical addition and multiplication by non-negative scalars, the union of all polars v° for $v \in V$ forms the *dual cone* P^* of P.

We may now formulate a topological version of the sandwich theorem (Theorem 3.1 in [13]) for linear functionals: Generalizing our previous notion we define an *extended superlinear functional* on P as a mapping

$$q: P \to \underline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

such that $q(\alpha a) = \alpha q(a)$ holds for all $a \in P$ and $\alpha \geq 0$ and

$$q(a+b) \geq q(a) + q(b) \quad \text{whenever} \quad q(a), q(b) > -\infty$$

(We set $\alpha + (-\infty) = -\infty$ for all $\alpha \in \mathbb{R} \cup \{-\infty\}$, $\alpha \cdot (-\infty) = -\infty$ for all $\alpha > 0$ and $0 \cdot (-\infty) = 0$ in this context.)

4.1 Sandwich Theorem (topological). *Let (P, V) be a locally convex cone, and let $v \in V$. For a sublinear functional $p : P \to \overline{\mathbb{R}}$ and an extended superlinear functional $q : P \to \underline{\mathbb{R}}$ there exists a linear functional $\mu \in v^\circ$ such that $q \leq \mu \leq p$ if and only if*

$$q(a) \leq p(b) + 1 \quad \textit{holds whenever} \quad a \leq b + v$$

Recall that every monotone linear functional μ on an ordered cone P gives rise to a uniform neighborhood $v = \{(a,b) \in P^2 \mid \mu(a) \leq \mu(b) + 1\}$ which in turn may be used to define a locally convex structure on P. Thus, the condition for p and q in Theorem 4.1 for some neighborhood v is necessary and sufficient for the existence of a monotone linear functional μ on P such that $q \leq \mu \leq p$.

Citing from [13] we mention a few corollaries. A set $C \subset P$ is called *increasing* resp. *decreasing*, if $a \in C$ whenever $c \leq a$ resp. $a \leq c$ for $a \in P$ and some $c \in C$. A convex set $C \subset P$ such that $0 \in C$ is called *left-absorbing* if for every $a \in P$ there are $c \in C$ and $\lambda \geq 0$ such that $\lambda c \leq a$.

4.2 Corollary. *Let P be an ordered cone. For a sublinear functional $p : P \to \overline{\mathbb{R}}$ there exists a monotone linear functional $\mu : P \to \overline{\mathbb{R}}$ such that $\mu \leq p$ if and only if p is bounded below on some increasing left-absorbing convex set $C \subset P$.*

An $\overline{\mathbb{R}}$-valued function f defined on a convex subset C of a cone P is called *convex* if

$$f(\lambda c_1 + (1-\lambda)c_2) \leq \lambda f(c_1) + (1-\lambda) f(c_2)$$

holds for all $c_1, c_2 \in C$ and $\lambda \in [0,1]$. Likewise, an $\underline{\mathbb{R}}$-valued function g on C is *concave* if

$$g(\lambda c_1 + (1-\lambda)c_2) \geq \lambda g(c_1) + (1-\lambda) g(c_2)$$

holds for all $c_1, c_2 \in C$ such that $g(c_1), g(c_2) > -\infty$ and $\lambda \in [0,1]$. An *affine* function $h : C \to \overline{\mathbb{R}}$ is both convex and concave. A variety of extension results for linear functionals may be derived from Theorem 4.1 in [13]. We cite:

4.3 Extension Theorem. *Let (P, V) be a locally convex cone, C and D non-empty convex subsets of P, and let $v \in V$. Let $p : P \to \overline{\mathbb{R}}$ be a sublinear and $q : P \to \underline{\mathbb{R}}$ an extended superlinear functional. For a convex function $f : C \to \overline{\mathbb{R}}$ and a concave function $g : D \to \underline{\mathbb{R}}$ there exists a monotone linear functional $\mu \in v^\circ$ such that*

$$q \leq \mu \leq p, \quad g \leq \mu \ \textit{on } D \quad \textit{and} \quad \mu \leq f \ \textit{on } C$$

if and only if

$$q(a) + \rho g(d) \leq p(b) + \sigma f(c) + 1 \quad \textit{holds whenever} \quad a + \rho d \leq b + \sigma c + v$$

for $a, b \in P$, $c \in C$, $d \in D$ and $\rho, \sigma > 0$ such that $q(a), \rho g(d) > -\infty$.

The generality of this result allows a wide range of special cases. If $g \equiv -\infty$, for example, we have to consider the condition of Theorem 4.3 only for $\rho = 0$, if $f \equiv +\infty$ only for $\sigma = 0$, and if both $g \equiv$

$-\infty$ and $f \equiv +\infty$, then Theorem 4.3 reduces to the previous Sandwich Theorem 4.1. Another case of particular interest occurs when $C = D$ and $f = g$ is an affine function, resp. a linear functional if C is a subcone of P. The latter, with the choice of $p(a) = +\infty$ and $q(a) = -\infty$ for all $0 \neq a \in P$ yields the Extension Theorem II.2.9 from [7]:

4.4 Corollary. *Let (C, V) be a subcone of the locally convex cone (P, V). Every continuous linear functional on C can be extended to a continuous linear functional on P; more precisely: For every $\mu \in v_C^\circ$ there is $\tilde{\mu} \in v_P^\circ$ such that $\tilde{\mu}$ coincides with μ on C.*

The range of all continuous linear functionals that are sandwiched between a given sublinear and an extended superlinear functional is described in Theorem 5.1 in [13].

4.5 Range Theorem. *Let (P, V) be a locally convex cone. Let p and q be sublinear and extended superlinear functionals on P and suppose that there is at least one linear functional $\mu \in P^*$ satisfying $q \leq \mu \leq p$. Then for all $a \in P$ we have*

$$\sup_{\mu \in P^*, q \leq \mu \leq p} \mu(a) = \sup_{v \in V} \inf\{p(b) - q(c) \mid b, c \in P, q(c) \in \mathbb{R}, a + c \leq b + v\}$$

For all $a \in P$ such that $\mu(a)$ is finite for at least one $\mu \in P^*$ satisfying $q \leq \mu \leq p$ we have

$$\inf_{\mu \in P^*, q \leq \mu \leq p} \mu(a) = \inf_{v \in V} \sup\{q(c) - p(b) \mid b, c \in P, p(b) \in \mathbb{R}, c \leq a + b + v\}$$

As another consequence of the Extension Theorem 4.3 we obtain the following result (Theorem 4.5 in [13]) about the separation of convex subsets by monotone linear functionals:

4.6 Separation Theorem. *Let C and D be non-empty convex subsets of a locally convex cone (P, V). Let $v \in V$ and $\alpha \in \mathbb{R}$. There exists a monotone linear functional $\mu \in v^\circ$ such that*

$$\mu(c) \leq \alpha \leq \mu(d) \quad \text{for all } c \in C \text{ and } d \in D$$

if and only if

$$\alpha\rho \leq \alpha\sigma + 1 \quad \text{whenever} \quad \rho d \leq \sigma c + v$$

for all $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$.

5. The weak preorder and the relative topologies

We also consider a (topological and linear) closure of the given order on a locally convex cone, called the weak preorder $\preccurlyeq$ which is defined as follows (see I.3 in [17]): We set

$$a \preccurlyeq b + v \quad \text{for } a, b \in P \text{ and } v \in V$$

if for every $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ such that $a \leq \gamma b + (1 + \varepsilon)v$, and set

$$a \preccurlyeq b$$

if $a \preccurlyeq b + v$ for all $v \in V$. This order is clearly weaker than the given order, that is $a \leq b$ or $a \leq b + v$ implies $a \preccurlyeq b$ or $a \preccurlyeq b + v$. Importantly, the weak preorder on a locally convex cone is entirely determined by its dual cone P^*, that is $a \preccurlyeq b$ holds if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in P^*$, and $a \preccurlyeq b + v$ if and only $\mu(a) \leq \mu(b) + 1$ for all $\mu \in v^\circ$ (Corollaries I.4.31 and I.4.34 in [17]). If endowed with the weak preorder (P, V) is again a locally convex cone with the same dual P^*.

While all elements of a locally convex cone are bounded below, they need not be bounded above. An element $a \in P$ is called *bounded (above)* (see [7], I.2.3) if for every $v \in V$ there is $\lambda > 0$ such that $a \leq \lambda v$. By $\mathcal{B}$ we denote the subcone of P containing all bounded elements. $\mathcal{B}$ is indeed a *face* of P, as $a + b \in \mathcal{B}$ for $a, b \in P$ implies that both $a, b \in \mathcal{B}$. Clearly all invertible elements of P are bounded, and bounded elements satisfy a modified version of the cancellation law (see [17], I.4.5), that is

$$(\mathrm{C}') \qquad a + c \preccurlyeq b + c \ \text{ for } a, b \in P \text{ and } c \in \mathcal{B} \ \text{ implies } \ a \preccurlyeq b$$

We quote Theorem I.3.3 from [17]:

5.1 Representation Theorem. *A locally convex cone (P,V) endowed with its weak preorder can be represented as a locally convex cone of $\overline{\mathbb{R}}$-valued functions on some set X, or equivalently as a locally convex cone of convex subsets of some locally convex ordered topological vector space.*

The previously introduced upper, lower and symmetric locally convex cone topologies for a locally convex cone (P,V) prove to be too restrictive for the concept of continuity of P-valued functions, since for unbounded elements even the scalar multiplication turns out to be discontinuous (see I.4 in [17]). This is remedied by using the coarser (but somewhat cumbersome) *relative* topologies on P instead. These topologies are defined using the weak preorder on P:

The *upper, lower and symmetric relative topologies* on a locally convex cone (P,V) are generated by the neighborhoods $v_\varepsilon(a)$, $(a)v_\varepsilon$ and $v_\varepsilon^s(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$, respectively, for $a \in P$, $v \in V$ and $\varepsilon > 0$, where

$$v_\varepsilon(a) = \{b \in P \mid b \leq \gamma a + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon\}$$
$$(a)v_\varepsilon = \{b \in P \mid a \leq \gamma b + \varepsilon v \text{ for some } 1 \leq \gamma \leq 1+\varepsilon\}$$

The relative topologies are locally convex but not necessarily locally convex cone topologies in the sense of Section 3 (for details see I.4 in [17]), since the resulting uniformity need not be convex. These topologies are generally coarser, but locally coincide on bounded elements with the given upper, lower and symmetric topologies on P and render the scalar multiplication (with scalars other than zero) continuous. The symmetric relative topology is known to be Hausdorff if and only if the weak preorder on P is antisymmetric (Proposition I.4.8 in [17]). If P is a locally convex topological vector space, then all of the above topologies coincide with the given topology.

6. Boundedness and connectedness components

The details for this section can be found in [16]. Two elements a and b of a locally convex cone (P,V) are *bounded relative to each other* if for every $v \in V$ there are $\alpha, \beta, \lambda, \rho \geq 0$ such that both

$$a \leq \beta b + \lambda v \quad \text{and} \quad b \leq \alpha a + \rho v$$

This notion defines an equivalence relation on P and its equivalence classes $\mathcal{B}_s(a)$ are called the *(symmetric) boundedness components* of P. Propositions 5.3, 5.4, 5.6 and 6.1 in [16] state:

6.1 Proposition. *The boundedness components of a locally convex cone (P,V) are closed for addition and multiplication by strictly positive scalars. They satisfy a version of the cancellation law, that is*

$$a + c \preccurlyeq b + c$$

for elements a, b and c of the same boundedness component implies that

$$a \preccurlyeq b.$$

6.2 Proposition. *The boundedness components of a locally convex cone (P,V) furnish a partition of P into disjoint convex subsets that are closed and connected in the symmetric relative topology. They coincide with the connectedness components of P.*

If the neighborhood system V consists of the positive multiples of a single neighborhood, P is locally connected and its connectedness components are also open.

7. Continuous linear operators

For cones P and Q a mapping $T : P \to Q$ is called a *linear operator* if

$$T(a+b) = T(a) + T(b) \quad \text{and} \quad T(\alpha a) = \alpha T(a)$$

hold for all $a, b \in P$ and $\alpha \geq 0$. If both P and Q are ordered, then T is called *monotone* if

$$a \leq b \quad \text{implies} \quad T(a) \leq T(b).$$

If both (P, V) and (Q, W) are locally convex cones, then T is said to be *(uniformly) continuous* if for every $w \in W$ one can find $v \in V$ such that

$$T(a) \leq T(b) + w \quad \text{whenever} \quad a \leq b + v$$

for $a, b \in P$. A set $\hat{T}$ of linear operators is called *equicontinuous* if the above condition holds for every $w \in W$ with the same $v \in V$ for all $T \in \hat{T}$. Uniform continuity for an operator implies monotonicity with respect to the global preorders on P and on Q that is: if

$$a \leq b + v \text{ for all } v \in V, \quad \text{then}$$
$$T(a) \leq T(b) + w \text{ for all } w \in W$$

In this context, a linear functional is a linear operator $\mu : P \to \overline{\mathbb{R}}$, and the above notion of continuity conforms to the preceding one (see Section 4). Moreover, for two continuous linear operators S and T from P into Q and for $\lambda \geq 0$, the sum $S + T$ and the multiple λT are again linear and continuous. Thus the continuous linear operators from P into Q again form a cone. The *adjoint operator* T^* of $T : P \to Q$ is defined by

$$\big(T^*(\nu)\big)(a) = \nu(T(a))$$

for all $\nu \in Q^*$ and $a \in P$. Clearly $T^*(\nu) \in P^*$, and T^* is a linear operator from Q^* to P^*; more precisely: If for $v \in V$ and $w \in W$ we have $T(a) \leq T(b) + w$ whenever $a \leq b + v$, then T^* maps w° into v°.

While some concepts from duality and operator theory of locally convex vector spaces may be readily transferred to the more general context of locally convex cones, others require a new approach and offer insights into a far more elaborate structure. The concept of completeness, for example, does not lend itself to a straightforward transcription. It is adapted to locally convex cones in [12] in order to allow a reformulation of the uniform boundedness principle for Fréchet spaces. The approach uses the notions of *internally bounded* subsets, *weakly cone complete* and *barreled* cones. These definitions turn out to be rather technical and we refrain from supplying the details. We cite the main result, which generalizes the classical uniform boundedness theorem:

7.1 Uniform Boundedness Theorem. *Let* (P, V) *and* (Q, W) *be locally convex cones, and let* $\hat{T}$ *be a family of u-continuous linear operators from* P *to* Q. *Suppose that for every* $b \in P$ *and* $w \in W$ *there is* $v \in V$ *such that for every* $a \in v(b) \cap (b)v$ *there is* $\lambda > 0$ *such that*

$$T(a) \leq T(b) + \lambda w \quad \textit{for all } T \in \hat{T}$$

If (P, V) *is barreled and* (Q, W) *has the strict separation property [that is,* (Q, W) *satisfies Theorem 4.6)], then for every internally bounded set* $\mathcal{B} \subset P$, *every* $b \in \mathcal{B}$ *and* $w \in W$ *there is* $v \in V$ *and* $\lambda > 0$ *such that*

$$T(a) \leq T(b) + \lambda w \quad \textit{for all } T \in \hat{T}$$

and all $a \in v(b') \cap (b'')v$ *for some* $b', b'' \in \mathcal{B}$.

8. Duality of cones and inner products

We excerpt and augment the following from Ch.II.3 in [7]: A *dual pair* (P, Q) consists of two ordered cones P and Q together with a bilinear map, i.e. a mapping

$$(a, b) \to \langle a, b \rangle : P \times Q \to \overline{\mathbb{R}}$$

which is linear in both variables and compatible with the order structures on both cones, satisfying

$$\langle a, y \rangle + \langle b, x \rangle \leq \langle a, x \rangle + \langle b, y \rangle \quad \text{whenever}$$
$$a \leq b \text{ and } x \leq y.$$

Let us denote by

$$P^+ = \{a \in P \mid 0 \leq a\} \quad \text{and}$$
$$Q^+ = \{a \in Q \mid 0 \leq a\}$$

the respective subcones of positive elements in P and Q. The above condition guarantees that all elements $x \in Q^+$, via $a \to \langle a, x \rangle$ define monotone linear functionals on P, and vice versa.

If we endow the dual cone P^* of a locally convex cone (P,V) with the canonical order

$$\mu \leq \nu \quad \text{if} \quad \nu = \mu + \sigma \ \text{ for some } \ \sigma \in P^*,$$

then all elements $\mu \in P^*$ are positive. With the evaluation as its canonical bilinear form, (P,P^*) forms a dual pair.

Dual pairs give rise to *polar topologies* in the following way: A subset X of Q^+ is said to be *σ-bounded below* if

$$\inf\{\langle a,x\rangle \mid x \in X\} > -\infty$$

for all $a \in P$. Every such subset $X \subset Q^+$ defines a uniform neighborhood $v_X \in P^2$ by

$$v_X = \{(a,b) \in P^2 \mid \langle a,x\rangle \leq \langle b,x\rangle + 1 \ \text{for all } x \in X\}$$

and any collection χ of σ-bounded below subsets of Q satisfying:

(P1) $\lambda X \in \chi$ whenever $X \in \chi$ and $\lambda > 0$.
(P2) For all $X, Y \in \chi$ there is some $Z \in \chi$ such that $X \cup Y \subset Z$.

defines a convex quasiuniform structure on P. If we denote the corresponding neighborhood system by $V_\chi = \{v_X \mid X \in \chi\}$, then (P, V_χ) becomes a locally convex cone. The polar v_X° of the neighborhood $v_X \in V_\chi$ consists of all linear functionals μ on P such that for $a, b \in P$

$$\langle a,x\rangle \leq \langle b,x\rangle + 1 \quad \text{for all } x \in X \quad \text{implies that} \quad \mu(a) \leq \mu(b) + 1.$$

All elements of $X \subset Q$, considered as linear functionals on P, are therefore contained in v_X°.

8.1 Examples. (a) Let χ be the family of all finite subsets of Q^+. The resulting polar topology on P is called the *weak*-topology $\sigma(P,Q)$*.

(b) Let (P,V) be a locally convex cone with the strict separation property (SP). Consider the dual pair (P,P^*) and the collection χ of the polars $v^\circ \subset P^*$ of all neighborhoods $v \in V$. The resulting polar topology on P coincides with the original one. This shows in particular that every locally convex cone topology satisfying (SP) may be considered as a polar topology.

Two specific topologies on Q, denoted $w(Q,P)$ and $s(Q,P)$, are of particular interest: Both are topologies of pointwise convergence for the elements of P considered as functions on Q with values in $\overline{\mathbb{R}}$. For $w(Q,P)$, $\overline{\mathbb{R}}$ is considered with its usual (one-point compactification) topology, whereas $+\infty$ is treated as an isolated point for $s(Q,P)$. A typical neighborhood for $x \in Q$, defined via a finite subset $A = \{a_1, \ldots, a_n\}$ of P, is given in the topology $w(Q,P)$ by

$$W_A(x) = \left\{ y \in Q \,\middle|\, \begin{array}{ll} |\langle a_i,y\rangle - \langle a_i,x\rangle| \leq 1, & \text{if } \langle a_i,x\rangle < +\infty \\ \langle a_i,y\rangle > 1, & \text{if } \langle a_i,x\rangle = +\infty \end{array} \right\}$$

and in the topology $s(Q,P)$ by

$$S_A(x) = \left\{ y \in Q \,\middle|\, \begin{array}{ll} |\langle a_i,y\rangle - \langle a_i,x\rangle| \leq 1, & \text{if } \langle a_i,x\rangle < +\infty \\ \langle a_i,y\rangle = +\infty, & \text{if } \langle a_i,x\rangle = +\infty \end{array} \right\}$$

In general, $s(Q,P)$ is therefore finer than $w(Q,P)$, but both topologies coincide if the bilinear form on $P \times Q$ attains only finite values.

In analogy to the Alaoglu-Bourbaki theorem in locally convex vector spaces (see [18], III.4), we obtain (Proposition 2.4 in [7]):

8.2 Theorem. *Let (P,V) be a locally convex cone. The polar v° of any neighborhood $v \in V$ is a compact convex subset of P^* with respect to the topology $w(P^*,P)$.*

Likewise, a Mackey-Arens type result is available for locally convex cones (Theorem 3.8 in [7]):

8.3 Theorem. *Let (P,Q) be a dual pair of ordered cones, and let $X \subset Q$ be the union of finitely many $s(Q,P)$-compact convex subsets of Q^+. Then for every linear functional $\mu \in v_X^\circ$ on P there is an element $x \in Q$ such that*

$$\mu(a) = \langle a, x \rangle \quad \textit{for all } a \in P \quad \textit{with} \quad \mu(a) < +\infty.$$

The last theorem applies is particular to the weak*-topology $\sigma(P, Q)$ which is generated by the finite subsets of Q.

An *inner product* on an ordered cone P may be defined as a bilinear form on $P \times P$ which is commutative and satisfies

$$2\langle a, b \rangle \leq \langle a, a \rangle + \langle b, b \rangle \quad \text{for all } a, b \in P$$

Investigations on inner products yield Cauchy-Schwarz and Bessel-type inequalities, concepts for orthogonality and best approximation, as well as an analogy for the Riesz representation theorem for continuous linear functionals. For details we refer to [14].

9. Extended algebraic operations

Example 2.1 (b) suggests that the scalar multiplication in a cone might be canonically extended for all scalars in $\mathbb{R}$ or $\mathbb{C}$, but only a weakened version of the distributive law holds for non-positive scalars. For details of the following we refer to [11]. Let $\mathbb{K}$ denote either the field of the real or the complex numbers, and

$$\Delta = \{\delta \in \mathbb{K} \mid |\delta| \leq 1\},$$
$$\text{resp.} \quad \Gamma = \{\gamma \in \mathbb{K} \mid |\gamma| = 1\}$$

the closed unit disc, resp. unit sphere in $\mathbb{K}$.

An ordered cone P is *linear over* $\mathbb{K}$ if the scalar multiplication is extended to all scalars in $\mathbb{K}$ and in addition to the requirements for an ordered cone satisfies

$$\alpha(\beta a) = (\alpha\beta)a \quad \text{for all } a \in P \text{ and } \alpha, \beta \in \mathbb{K}$$
$$\alpha(a + b) = \alpha a + \alpha b \quad \text{for all } a, b \in P \text{ and } \alpha \in \mathbb{K}$$
$$(\alpha + \beta)a = \alpha a + \beta a \quad \text{for all } a \in P \text{ and } \alpha, \beta \in \mathbb{K}$$

It is necessary in this context to distinguish carefully between the additive inverse $-a$ of an element $a \in P$ which may exist in P, and the element $(-1)a \in P$. Both need not coincide.

We define the *modular order* $\preccurlyeq_m$ for elements $a, b \in P$ by

$$a \preccurlyeq_m b \quad \text{if} \quad \gamma a \leq \gamma b \text{ for all } \gamma \in \Gamma$$

The basic properties of an order relation are easily checked. Likewise the relation $\preccurlyeq_m$ is seen to be compatible with the extended algebraic operations in P, i.e.

$$a \preccurlyeq_m b \quad \text{implies} \quad \lambda a \preccurlyeq_m \lambda b \quad \text{and} \quad a + c \preccurlyeq_m b + c$$

for all $\lambda \in \mathbb{K}$ and $c \in P$. Obviously

$$a \preccurlyeq_m b \quad \text{implies that} \quad a \leq b.$$

Indeed, our version of the distributive law entails that

$$(\alpha + \beta)a \preccurlyeq_m \alpha a + \beta a$$
holds for all $a \in P$ and $\alpha, \beta \in \mathbb{K}$.

Using the modular order we define an equivalence relation $\sim_m$ on P by

$$a \sim_m b \quad \text{if} \quad a \preccurlyeq_m b \text{ and } b \preccurlyeq_m a$$

An element $a \in P$ is called $\widetilde{m}$*-invertible* if there is $b \in P$ such that $a + b \sim_m 0$. Any two $\widetilde{m}$-inverses of the same element a are equivalent in the above sense. We summarize a few observations (Lemma 2.1 in [11]):

9.1 Lemma. *Let P be an ordered cone that is linear over $\mathbb{K}$. Then*

(a) $\alpha 0 = 0$ *for all* $\alpha \in \mathbb{K}$.
(b) $0 \preccurlyeq_m a + (-1)a$ *for all* $a \in P$.
(c) *If $a \in P$ is $\widetilde{m}$-invertible, then $(\alpha + \beta)a \sim_m \alpha a + \beta a$ holds for all $\alpha, \beta \in \mathbb{K}$, and $(-1)a \sim_m b$ for all $\widetilde{m}$-inverses b of a.*
(d) *If both $a, b \in P$ are $\widetilde{m}$-invertible, then $a \preccurlyeq_m b$ implies $a \sim_m b$.*

If (P, V) is a locally convex cone and P is linear over $\mathbb{K}$, then the neighborhoods $v \in V$ and the modular order on P give rise to corresponding *modular neighborhoods* $v_m \in V_m$ in the following way: For $a, b \in P$ and $v \in V$ we define

$$a \preccurlyeq_m b + v_m$$

if $\gamma a \preccurlyeq_m \gamma b + v$ for all $\gamma \in \Gamma$. Clearly $a \preccurlyeq_m b + v_m$ implies that $\lambda a \preccurlyeq_m \lambda b + |\lambda| v_m$ for all $\lambda \in \mathbb{K}$. We denote the system of modular neighborhoods on P by V_m. If we require that every element $a \in P$ is also bounded below with respect to these modular neighborhoods, i.e. if for every $v \in V$ there is $\lambda > 0$ such that

$$0 \leq \gamma a + \lambda v \quad \text{for all } \gamma \in \Gamma,$$

then (P, V_m) with the modular order is again a locally convex cone. In this case we shall say that (P, V) is a *locally convex cone over* $\mathbb{K}$. The respective (upper, lower and symmetric) modular topologies on P are finer than those resulting from the original neighborhoods in V.

9.2 Examples. (a) Let $P = \overline{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$ be endowed with the usual algebraic operations, in particular $a + \infty = \infty$ for all $a \in \overline{\mathbb{K}}$, $\alpha \cdot \infty = \infty$ for all $0 \neq \alpha \in \mathbb{K}$ and $0 \cdot \infty = 0$. The order on $\overline{\mathbb{K}}$ is defined by

$$a \leq b \quad \text{if } b = \infty \text{ or } \Re(a) \leq \Re(b).$$

With the neighborhood system $V = \{\varepsilon > 0\}$, $\overline{\mathbb{K}}$ is a full locally convex cone. It is easily checked that $\overline{\mathbb{K}}$ is linear over $\mathbb{K}$. The modular order on $\overline{\mathbb{K}}$ is identified as $a \preccurlyeq_m b$ if either $b = \infty$ or $a = b$. For $v = \varepsilon \in V$ we have $a \preccurlyeq_m b + v_m$ if either $b = \infty$ or $|a - b| \leq \varepsilon$.

(b) We augment our Example 3.1 (c) as follows: Let $(E, \leq)$ be a locally convex ordered topological vector space over $\mathbb{K}$. For $A \in P = Conv(E)$ we define the multiplication by *any* scalar $\alpha \in \mathbb{K}$ by

$$\alpha A = \{\alpha a |\ a \in A\}$$

for $\alpha \in \mathbb{K}$ and $A \in P$, and the addition and order as in 3.1 (c), that is

$$A \leq B \quad \text{if} \quad A \subset \downarrow B$$

Thus P is linear over $\mathbb{K}$. Considering the modular order on P, for $A \in P$ we denote by

$$\downarrow_m A = \bigcap_{\gamma \in \Gamma} (\bar{\gamma} \downarrow (\gamma A))$$

(for $\mathbb{K} = \mathbb{R}$ this is just the order interval generated by A). Thus

$$A \preccurlyeq_m B \quad \text{if} \quad A \subset \downarrow_m B$$

As in 3.1 (c), the abstract neighborhood system in P is given by a basis $V \subset P$ of closed absolutely convex neighborhoods of the origin in E. Every element $A \in P$ is seen to be m-bounded below, thus fulfilling the last requirement for a locally convex cone over $\mathbb{K}$.

The case $E = \mathbb{K}$ with the order from 9.2 (a), i.e. $a \leq b$ if $\Re(a) \leq \Re(b)$, is of particular interest for the investigation of linear functionals: For $A, B \in Conv(\mathbb{K})$ we have $A \leq B$ if $\sup\{\Re(a)|\ a \in A\} \leq \sup\{\Re(b)|\ b \in B\}$ and $A \preccurlyeq_m B$ if $A \subset B$. For $\varepsilon > 0$ the neighborhood $\varepsilon\Delta \in V$ is determined by

$$A \leq B \oplus \varepsilon\Delta$$
$$\text{if } \sup\{\Re(a)|\ a \in A\} \leq \sup\{\Re(b)|\ b \in B\} + \varepsilon,$$

and

$$A \preccurlyeq_m B \oplus \varepsilon\Delta_m \quad \text{if} \quad A \subset B \oplus \varepsilon\Delta$$

(c) Let P consist of all $\overline{\mathbb{R}}$-valued functions f on $[-1, +1]$ that are uniformly bounded below and satisfy $0 \leq f(x) + f(-x)$ for all $x \in [-1, +1]$. Endowed with the pointwise addition and multiplication by non-negative scalars, the order $f \leq g$ if $f(x) \leq g(x)$ for all $0 \leq x \leq 1$, and the neighborhood system V consisting of the (strictly) positive constants, P is a full locally convex cone. We may extend the scalar multiplication to negative reals α and $f \in P$ by

$$(\alpha f)(x) = (-\alpha) f(-x)$$

for all $x \in [-1, +1]$. Thus P is seen to be linear over $\mathbb{R}$. The modular order on P is the pointwise order on the whole interval $[-1, +1]$.

For a locally convex cone over $\mathbb{K}$ we shall denote by $\mathcal{B}_m$ the subcone of all *m-bounded* elements, i.e. those elements $a \in P$ such that for every $v \in V$ there is $\lambda > 0$ such that $a \preccurlyeq_m \lambda v_m$. Clearly $\mathcal{B}_m \subset . \mathcal{B}$. We cite Theorem 2.3 from [11]:

9.3 Theorem. *Every locally convex cone (P, V) can be embedded into a locally convex cone $(\tilde{P}, V)$ over $\mathbb{K}$. The embedding is linear, one-to-one and preserves the global preorder and the neighborhoods of P. All bounded elements $a \in P$ are mapped onto m-bounded elements of $\tilde{P}$ and are $\tilde{m}$-invertible in $\tilde{P}$.*

Let (P, V) be a locally convex cone over $\mathbb{K}$. Endowed with the corresponding modular neighborhood system, (P, V_m) is again a locally convex cone. We denote the dual cone of (P, V_m) by P_m^* and refer to it as the *modular dual* of P. As continuity with respect to the given topology implies continuity with respect to the modular topology we have $P^* \subset P_m^*$. By v_m° we denote the (modular) polar of the neighborhood $v_m \in V_m$, i.e. the set of all linear functionals $\mu \in P_m^*$ such that

$$\mu(a) \le \mu(b) + 1 \text{ holds whenever } a \preccurlyeq_m b + v_m$$

Monotone linear functionals in $\mu : P \to \overline{\mathbb{R}}$ are required to be homogeneous only with respect to the multiplication by positive reals. For negative reals $\alpha < 0$ the relation $\alpha a + (-\alpha)a \ge 0$ yields $\mu(\alpha a) \ge \alpha\mu(a)$. But for complex numbers α in general we fail to recognize any obvious relation between $\mu(\alpha a)$ and $\alpha\mu(a)$. This may be remedied, at least for a large class of functionals in P_m^*, by the following procedure: An element $a \in P$ is called *m-continuous* if the mapping

$$\gamma \to \gamma a : \Gamma \to P$$

is uniformly continuous with respect to the upper topology on P, i.e. if for every $v \in V$ there is $\varepsilon > 0$ such that $\gamma a \le \gamma' a + v$ holds for all $\gamma, \gamma' \in \Gamma$ satisfying $|\gamma - \gamma'| \le \varepsilon$. For $\mathbb{K} = \mathbb{R}$ this condition is obviously void. For $\mathbb{K} = \mathbb{C}$, however, the m-continuous elements form a subcone of P which we shall denote by C_m. Obviously $\mathcal{B}_m \subset C_m$. A functional $\mu \in P_m^*$ is called *regular* if

$$\mu(a) = \sup\{\mu(c) |\ c \in C_m, c \preccurlyeq_m a\}$$

holds for all $a \in P$. For $\mathbb{K} = \mathbb{R}$, of course, as all elements $a \in P$ are m-continuous, every $\mu \in P_m^*$ is regular. For a regular linear functional $\mu \in P_m^*$ and every $a \in P$ we may define a corresponding set-valued function $\mu_c : P \to Conv(\mathbb{K})$ by

$$\mu_c(a) = \{\alpha \in \mathbb{K} |\ \Re(\gamma\alpha) \le \mu(\gamma a) \text{ for all } \gamma \in \mathbb{K}\}$$

The regularity of μ entails (see [11]) that $\mu_c(a)$ is non-empty, closed and convex in $\mathbb{K}$, and that

$$\mu(\gamma a) = \sup\{\Re(\gamma\alpha) |\ \alpha \in \mu_c(a)\}$$

holds for all $\gamma \in \mathbb{K}$. The latter shows in particular that the correspondence between μ and μ_c is one-to-one. For $\mathbb{K} = \mathbb{R}$ the values of μ_c are closed intervals in $\mathbb{R}$; more precisely:

$$\mu_c(a) = [-\mu\big((-1)a\big), \mu(a)] \cap \mathbb{R}.$$

The mapping $\mu_c : P \to Conv(\mathbb{K})$ is additive and homogeneous with respect to the multiplication by all scalars in $\mathbb{K}$. More precisely:

9.4 Lemma. *Let $\mu : P \to \overline{\mathbb{R}}$ be a regular monotone linear functional. For $\mu_c : P \to Conv(\mathbb{K})$ the following hold:*

(a) $\mu_c(a)$ is a non-empty closed convex subset of $\mathbb{K}$.
(b) $\mu_c(a + b) = \mu_c(a) \oplus \mu_c(b)$ for all $a, b \in P$.
(c) $\mu_c(\alpha a) = \alpha\mu_c(a)$ for all $a \in P$ and $\alpha \in \mathbb{K}$.
(d) If $a \in P$ is $\tilde{m}$-invertible then $\mu_c(a)$ is a singleton subset of $\mathbb{K}$.
(e) μ_c is continuous with respect to the modular topologies on P and $Conv(\mathbb{K})$; more precisely: if $\mu \in v_m^\circ$ then, for $a, b \in P$,

$$a \preccurlyeq_m b + v_m \quad \textit{implies that} \quad \mu_c(a) \subset \mu_c(b) \oplus \Delta,$$

where Δ denotes the closed unit disc in $\mathbb{C}$.

9.5 Examples. Reviewing our Example 9.2 (b), i.e. the locally convex cone $P = Conv(E)$ over $\mathbb{K}$, where $(E, \leq)$ denotes a locally convex ordered topological vector space, we realize that for every $\mathbb{K}$-valued continuous linear functional f on E, the mapping $\mu: P \to \overline{\mathbb{R}}$ such that

$$\mu(A) = \sup\{\Re(f(a)) \mid a \in A\}$$

is linear, an element of P_m^* and obviously regular. The corresponding set-valued functional $\mu_c : P \to Conv(\mathbb{K})$ is given by

$$\mu_c(A) = f(A) = \{f(a) \mid a \in A\}.$$

However, in the complex case, even for $E = \mathbb{C}$, one can find examples of non-regular linear functionals in P_m^*.

However, in the complex case, even for $E = \mathbb{C}$, one can find examples of non-regular linear functionals in P_m^*.

It is possible to construct a decomposition for regular functionals $\mu \in P_m^*$ into functionals in P^*. In a locally convex ordered topological vector space over $\mathbb{R}$ every continuous linear functional may be expressed as a difference of two positive ones (see [18], IV.3.2). A similar decomposition is available in the complex case. The more general setting of locally convex cones, however, requires the use of Riemann-Stieltjes type integrals instead of sums. In this instance we refrain from supplying the detailed arguments and notations for this rather technical procedure. They may be found in [11]. The main result is:

9.6 Theorem. *Let (P, V) be a locally convex cone over $\mathbb{K}$. For every regular linear functional $\mu \in P_m^*$ there exists a P^*-valued m-integrating family $(\vartheta_E)_{E\in\mathbb{R}}$ on the unit circle Γ in $\mathbb{C}$ such that*

$$\mu = \int_\Gamma \gamma \, d\vartheta$$

In the case of a locally convex cone over $\mathbb{R}$, where $\Gamma = \{-1, +1\}$, this result simplifies considerably. Every linear functional $\mu \in P_m^*$ is regular then, and the integral representation in Theorem 7.6 reduces to a sum of two functionals.

9.7 Corollary. *Let (P, V) be a locally convex cone over $\mathbb{R}$. For every linear functional $\mu \in P_m^*$ there exist $\mu_1, \mu_2 \in P^*$ such that*

$$\mu(a) = \mu_1(a) + \mu_2((-1)a) \quad \textit{for all } a \in P.$$

10. Application: Korovkin type approximation

Locally convex cones provide a suitable setting for a rather general approach to Korovkin type theorems, an extensively studied field in abstract approximation theory. For a detailed survey on this subject we refer to [2]. Approximation schemes may often be modeled by sequences (or nets) of linear operators. For a sequence $(T_n)_{n\in\mathbb{N}}$ of positive linear operators on $C([0,1])$, Korovkin's theorem (see [8]) states that $T_n(f)$ converges uniformly to f for every $f \in C([0,1])$, whenever $T_n(g)$ converges to g for the three test functions $g = 1, x, x^2$. This result was subsequently generalized for different sets of test functions g and different topological spaces X replacing the interval $[0,1]$. Classical examples include the Bernstein operators and the Fejér sums which provide approximation schemes by polynomials and trigonometric polynomials, respectively. Further generalizations investigate the convergence of certain classes of linear operators on various domains, such as positive operators on topological vector lattices, contractive operators on normed spaces, multiplicative operators on Banach algebras, monotone operators on set-valued functions, monotone operators with certain restricting properties on spaces of stochastic processes, etc. Typically, for a subset M of a domain L one tries to identify all elements $f \in L$ such that

$$T_\alpha(g) \to g \ \text{ for all } \ g \in M \quad \text{implies that} \quad T_\alpha(f) \to f,$$

whenever $(T_\alpha)_{\alpha\in A}$ is an equicontinuous net (generalized sequence) in the restricted class of operators on L. Locally convex cones allow a unified approach to most of the above mentioned cases. Various restrictions on classes of operators may be taken care of by the proper choice of domains and their topologies alone and approximation results may be obtained through the investigation of continuous linear operators between locally convex cones. We proceed to outline a few results that may be found in Chapters III and IV of [7]:

Let Q be a subcone of the locally convex cone (P,V). The element $a \in P$ is said to be Q-*superharmonic* in $\mu \in P^*$ if $\mu(a)$ is finite and if for all $\nu \in P^*$,

$$\nu(b) \leq \mu(b) \text{ for all } b \in Q \quad \text{implies that} \quad \nu(a) \leq \mu(a)$$

This notation is derived from potential theory. We cite Theorem III.1.3 from [7] which is an immediate corollary to our Range Theorem 4.5 with the following insertions: We choose $q(a) = -\infty$ for all $a \neq 0$ and $p(a) = \mu(a)$ for $a \in Q$, otherwise $p(a) = +\infty$, and obtain:

10.1 Sup-Inf-Theorem. *Let Q be a subcone of the locally convex cone (P,V). Let $a \in P$ and $\mu \in P^*$ such that $\mu(a)$ is finite. Then a is Q-superharmonic in μ if and only if*

$$\mu(a) = \sup_{v \in V} \inf\{\mu(b) \mid b \in Q, a \leq b + v\}.$$

We shall cite only a simplified version of the main Convergence Theorem IV.1.13 in [7] for nets of linear operators on a locally convex cone. It is however sufficient to derive the classical results for Korovkin type approximation processes. For a net $(a_\alpha)_{\alpha \in A}$ in P we shall denote $a_\alpha \uparrow b$ if $(a_\alpha)_{\alpha \in A}$ converges towards $b \in P$ with respect to the upper topology, i.e. if for every $v \in V$ there is α_0 such that

$$a_\alpha \leq b + v \quad \text{for all } \alpha \geq \alpha_0.$$

10.2 Convergence Theorem. *Let Q be a subcone of the locally convex cone (P,V). Suppose that for every $v \in V$ the element $a \in P$ is Q-superharmonic in all functionals of the $w(P^*,P)$-closure of the set of extreme points of v°. Then for every equicontinuous net $(T_\alpha)_{\alpha \in A}$ of linear operators on P*

$$T_\alpha(b) \uparrow b \text{ for all } b \in Q \quad \text{implies that} \quad T_\alpha(a) \uparrow a.$$

Let us mention just one of the many well-known Korovkin type theorems that may be derived using Theorems 10.1 and 10.2: Let X be a locally compact Hausdorff space, $P = C_0(X)$ the space of all continuous real-valued functions on X that vanish at infinity, and let V consist of all positive constant functions. With the pointwise order and algebraic operations, (P,V) is a locally convex cone. Continuous linear operators on P are monotone and bounded with respect to the norm of uniform convergence on $C_0(X)$. The extreme points of polars of neighborhoods are just the non-negative multiples of point evaluations. Finally, convergence $f_\alpha \to f$ for a net of functions in $C_0(X)$ in the uniform topology means that both $f_\alpha \uparrow f$ and $(-f_\alpha) \uparrow (-f)$. We obtain a result due to Bauer and Donner [4]:

10.3 Theorem. *Let X be a locally compact Hausdorff space, and let M be a subset of $C_0(X)$. For a function $f \in C_0(X)$ the following are equivalent:*

(a) For every equicontinuous net $(T_\alpha)_{\alpha \in A}$ of positive linear operators on $C_0(X)$

$$T_\alpha(g) \to g \text{ for all } g \in M \quad \text{implies that} \quad T_\alpha(f) \to f$$

(Convergence is meant with respect to the topology of uniform convergence on X.)

(b) For every $x \in X$

$$f(x) = \sup_{\varepsilon>0} \inf\left\{ g(x) \,\middle|\, \begin{array}{c} g \in \mathrm{span}(M), \\ f \leq g + \varepsilon \end{array} \right\} = \inf_{\varepsilon>0} \sup\left\{ g(x) \,\middle|\, \begin{array}{c} g \in \mathrm{span}(M), \\ g \leq f + \varepsilon \end{array} \right\}$$

(c) For every $x \in X$ and for every bounded positive regular Borel measure μ on X

$$\mu(g) = g(x) \text{ for all } g \in M \quad \text{implies that} \quad \mu(f) = f(x)$$

The General Convergence Theorem IV.1.13 in [7] allows a far wider range of applications, including quantitative estimates for the order of convergence for the approximation processes modeled by sequences or nets of linear operators.

11. Application: Topological integration theory

A rather general approach to topological integration theory using locally convex cones is established in [10]. It utilizes techniques originally developed for Choquet theory. Continuous linear functionals on a given locally convex cone P are called *integrals* if they are minimal, resp. maximal with respect to certain subcones of P. Their properties resemble those of Radon measures on locally compact spaces. They satisfy convergence theorems corresponding to Fatou's Lemma and Lebesgue's theorem about bounded convergence. Depending on the choice of the determining subcones of P, one obtains a wide variety of applications, including classical integration theory on locally compact spaces (see [5]), Choquet theory about integral representation (see [1]), H-integrals on H-cones in abstract potential theory and monotone functionals on cones of convex sets. We shall outline some of the main concepts without supplying details and proofs which may be found in [10]:

Let (P, V) be a full locally convex cone, L and U two subcones of P. L is supposed to be a full cone, whereas all elements of U are supposed to be bounded. The following two conditions hold:

(U) *For all $a \in P$, $l \in L$, $u \in U$ such that $u \leq a + l$ and for every $v \in V$ there is $u' \in U$ such that $u' \leq a + v$ and $u \leq u' + l + v$.*

(L) *For all $a \in P$, $l \in L$, $u \in U$ such that $a + u \leq l$ and for every $v \in V$ there is $l' \in L$ such that $a \leq l'$ and $l' + u \leq l + v$.*

For linear functionals $\mu, \nu \in P^*$ we set

$$\mu \preccurlyeq \nu \quad \text{if} \quad \mu(l) \leq \nu(l) \text{ for all } l \in L \quad \text{and} \quad \mu(u) \geq \nu(u) \text{ for all } u \in U.$$

We write $\mu \sim \nu$ if both $\mu \preccurlyeq \nu$ and $\nu \preccurlyeq \mu$, i.e. if the functionals μ and ν coincide on U and L. *Integrals on P* are the minimal functionals in this order and (P, L, U) is called an *integration cone*.

11.1 Theorem. *Let (P, L, U) be an integration cone.*

(a) For every continuous linear functional $\mu_0 \in P^*$ there is an integral μ on P such that

$$\mu(l) \leq \mu_0(l) \text{ for all } l \in L \quad \text{and} \quad \mu(u) \geq \mu_0(u) \text{ for all } u \in U.$$

(b) The linear functional $\mu \in P^*$ is an integral if and only if

$$\mu(l) = \inf_{v\in V} \sup\{\mu(u) \mid u \leq l + v, u \in U\} \text{ for all } l \in L,$$

and

$$\mu(u) = \inf\{\mu(l) \mid u \leq l, l \in L\} \text{ for all } u \in U.$$

An element $a \in P$ is said to be *μ-integrable* with respect to an integral μ if

$$\nu \sim \mu \quad \text{implies that} \quad \nu(a) = \mu(a)$$

for all $\nu \in P^*$. For a given integral μ on P the μ-integrable elements form a subcone of P that contains both L and U.

11.2 Theorem. *Let μ be an integral on P. The element $a \in P$ is μ-integrable if and only if*

$$\inf_{v\in V} \sup\{\mu(u) \mid u \leq a + v, u \in U\} = \inf\{\mu(l) \mid a \leq l, l \in L\}.$$

For a Lebesgue-type convergence theorem we require a subset of special integrals that correspond to the point evaluations in classical integration theory. In this vein, for a neighborhood $v \in V$ we define the *integral boundary relative to v* to be the set Δv of all integrals δ on P such that

$\delta(v) < +\infty$, satisfying the following property: If for any two integrals μ_1, μ_2 on P we have

$$\delta(v) = (\mu_1 + \mu_2)(v) \quad \text{and}$$
$$\delta(u) \leq (\mu_1 + \mu_2)(u) \ \text{ for all } \ u \in U$$

then there are $\lambda_1, \lambda_2 \geq 0$ such that $\mu_1 \sim \lambda_1 \delta$ and $\mu_2 \sim \lambda_2 \delta$. For a neighborhood $v \in V$ we shall say that a subset A of P is *uniformly v-dominated* if there is $\rho \geq 0$ such that $a \leq \rho v$ for all $a \in A$.

We formulate the main convergence result (Theorem 4.3 in [11]) which is modeled after the Bishop de-Leeuw theorem from Choquet theory.

11.3 Theorem. *Let μ be an integral on the integration cone (P, L, U). For a neighborhood $v \in V$ let $(a_n)_{n \in \mathbb{N}}$ be a uniformly v-dominated sequence of μ-integrable elements in P. If*

$$\limsup_{n \in \mathbb{N}} \delta(a_n) \leq \delta(v)$$

for all $\delta \in \Delta v$, then

$$\limsup_{n \in \mathbb{N}} \mu(a_n) \leq \mu(v).$$

For detailed arguments in the following examples we refer to Examples 1.1 and 3.13 in [10].

11.4 Examples. (a) This example models topological integration theory on a compact Hausdorff space X as presented in [5]: Let P be the cone of all bounded below $\overline{\mathbb{R}}$-valued functions on X, endowed with the pointwise algebraic operations and order, and let V consist of all strictly positive constant functions on X. Then (P, V) is a full locally convex cone. We choose for L the subcone of all $\overline{\mathbb{R}}$-valued lower semicontinuous functions and for U all real-valued upper semicontinuous functions in P. As required, $V \subset L$, and all functions in U are bounded. For an integral $\mu \in P^*$, condition 11.1 (b) implies that

$$\mu(l) = \sup\{\mu(c) | \ c \leq l, c \in C(X)\}$$
$$\text{for all } \ l \in L$$

and

$$\mu(u) = \inf\{\mu(c) | \ u \leq c, c \in C(X)\}$$
$$\text{for all } \ u \in U.$$

Following Theorem 11.2, a function $f \in P$ is μ-integrable if and only if

$$\sup\{\mu(u) | \ u \leq f, u \in U\}$$
$$= \inf\{\mu(l) | \ f \leq l, l \in L\}.$$

The integrals of this theory, therefore are the positive Radon measures on the compact space X, and the above notion of integrability coincides with the usual one (see [5], IV.4, Théorème 3), except for the fact that we allow integrals to take the value $+\infty$. Theorem 11.1 (a) implies that every positive linear functional on $C(X)$ permits an extension to a positive Radon measure on X, which is the result of the Riesz Representation Theorem. For a neighborhood $v \in V$ the integral boundary relative to v consists of positive multiples of point evaluations in X. Thus Theorem 11.3 yields Lebesgue's convergence theorem. The adaptation of this example for a locally compact Hausdorff space X is rather more technical and may be found in [10], Example 3.13 (c).

(b) Let X be a compact convex subset of a locally convex Hausdorff space, and let (P, V) be as in (a). We choose the subcone of all $\overline{\mathbb{R}}$-valued lower semicontinuous concave functions for L and the real-valued upper semicontinuous convex functions for U. As the elements of the dual cone P^* of P when restricted to $C(X)$ are positive Radon measures on X, our integrals on P are just the usual maximal representation measures from classical Choquet theory. The μ-integrable elements of P include all continuous functions on X. Theorem 11.2 yields Mokobodzki's characterization of maximal measures in Choquet theory (Proposition 1.4.5 in [1]). The subspace $U \cap L$ consists of the continuous affine functions on X, and Theorem 11.1 (a) implies that every positive linear functional on this subspace (i.e. a positive multiple of a point evaluation on X) may be represented by such a maximal measure. Moreover, for every neighborhood $v \in V$, the integral boundary Δv consists of positive multiples of evaluations in the extreme points of

X, hence Theorem 11.3 recovers the Bishop de-Leeuw theorem from classical Choquet theory about the support of maximal measures.

(c) Let $(P = Conv(E), V)$ be the full locally convex cone introduced in Example 3.1 (c). We choose $L = P$ and for U the subcone of P of all singleton subsets of the space E. Following Theorem 11.2 every integral μ on P is already determined by its values on the subcone U, that is by a monotone continuous linear functional μ_0 in the usual dual E' of the locally convex ordered topological vector space E; that is

$$\mu(A) = \sup\{\mu_0(a) |\ a \in A\}$$

for every $A \in P$. This describes a one-to-one correspondence between the monotone functionals in E' and the integrals on P. For a neighborhood $v \in V$ the integral boundary relative to v consists of those integrals on P that are induced by positive multiples of the extreme points of the usual polar of v in E'.

References

[1] E. M. Alfsen, "Compact convex sets and boundary integrals," *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **1971**, vol. 57, Springer Verlag, Heidelberg-Berlin-New York.

[2] F. Altomare and M. Campiti, "Korovkin type approximation theory and its applications," *Gruyter Studies in Mathematics*, **1994**, vol. 17, Walter de Gruyter, Berlin-New York.

[3] B. Anger and J. Lembcke, "Hahn-Banach type theorems for hypolinear functionals," *Math. Ann.*, **1974**, 209, 127-151.

[4] H. Bauer and K. Donner, "Korovkin approximation in $C_0(X)$," *Math. Ann.*, **1978**, 236, 225-237.

[5] N. Bourbaki, *Éléments de Mathématique*, Fascicule III, Livre VI, Intégration, **1965**, Hermann, Paris.

[6] B. Fuchssteiner and W. Lusky, "Convex cones," North Holland Math. Studies, **1981**, vol.56.

[7] K. Keimel and W. Roth, "Ordered cones and approximation," Lecture Notes in Mathematics, 1517, **1992**, Springer Verlag, Heidelberg-Berlin-New York.

[8] P.P. Korovkin, "Linear operators and approximation theory," Russian Monographs and Texts on Advanced Mathematics, vol. III, **1960**, Gordon and Breach, New York.

[9] L. Nachbin, *Topology and Order*, **1965**, Van Nostrand, Princeton.

[10] W. Roth, "Integral type linear functionals on ordered cones," *Trans. Amer. Math. Soc.*, **1996**, vol. 348, no. 12, 5065-5085.

[11] W. Roth, "Real and complex linear extensions for locally convex cones," *Journal of Functional Analysis*, **1997**, vol. 151, no. 2, 437-454.

[12] W. Roth, "A uniform boundedness theorem for locally convex cones," *Proc. Amer. Math. Soc.*, **1998**, vol. 126, no. 7, 83-89.

[13] W. Roth, "Hahn-Banach type theorems for locally convex cones," *Journal of the Australian Math. Soc.* (Series A) 68, **2000** no. 1, 104-125.

[14] W. Roth, "Inner products on ordered cones," *New Zealand Journal of Mathematics*, **2001**, 30, 157-175.

[15] W. Roth, "Separation properties for locally convex cones," *Journal of Convex Analysis*, **2002**, vol. 9, No. 1, 301-307.

[16] W. Roth, "Boundedness and connectedness components for locally convex cones," *New Zealand Journal of Mathematics*, **2005**, 34, 143-158.

[17] W. Roth, "Operator-valued measures and integrals for cone-valued functions," Lecture Notes in Mathematics, vol. 1964, **2009**, Springer Verlag, Heidelberg-Berlin-New York.

[18] H.H. Schäfer, "Topological vector spaces," **1980**, Springer Verlag, Heidelberg-Berlin-New York.