Maximum boundaries for cones of continuous functions on a compact space and integral representations for linear functionals

Foo Chui Chen and Walter Roth*

Department of Mathematical Sciences, Faculty of Science, Universiti Brunei Darussalam, Jalan Tungku Link, Gadong, BE 1410, Brunei Darussalam

*corresponding author email: walter.roth@ubd.edu.bn

Abstract

We present a simplified and easily accessible approach to the integral representation for continuous linear functionals on a cone of continuous real-valued functions on a compact set. The measures defining these integrals are supported by the maximum boundary of the respective cones.

Index Terms: spaces and cones of continuous functions, integral representation

1. Introduction

The concept of a maximum boundary for an algebra of continuous functions on a compact space was first proposed by Georgii Šilov in 1964 [6]. It was later generalized to vector spaces of continuous functions not necessarily closed for multiplication using rather demanding and complicated techniques from Choquet theory (see [1], [4] and [2]). These also generate our results concerning integral representations for continuous linear functionals on these spaces. We offer a much simplified and more easily accessible approach in this paper while also generalizing the concepts from linear spaces to cones of continuous functions.

2. Maximum Boundaries

Let $X$ be a compact Hausdorff space and $C(X)$ the Banach space of all continuous functions on $X$ endowed with the maximum norm, that is

$$\|f\| = \max\{|f(x)| \mid x \in X\}.$$  

for $f \in C(X)$. A non-empty subset $H$ of $C(X)$ is a called a subcone of $C(X)$ if

$$f + g \in H \quad \text{and} \quad \alpha f \in H,$$

whenever $f, g \in H$, and $\alpha \geq 0$. Linear subspaces are of course subcones in this sense. For a function $f \in C(X)$ and a closed subset $Y$ of $X$ we abbreviate

$$\max(f, Y) = \max\{|f(x)| \mid x \in Y\}.$$  

Given a subcone $H$ of $C(X)$, a closed subset $Y$ of $X$ is called a (maximum) boundary for $H$ if

$$\max(f, Y) = \max(f, X)$$

holds for all $f \in H$, that is if all functions in $H$ attain their maximum value on $Y$. If $H$ is indeed a linear subspace of $C(X)$, then the functions in $H$ also take their minimum values on $Y$, since a function $f \in H$ takes its minimum value where $-f \in H$ takes its maximum value. We shall use Zorn's Lemma to prove that for every subcone of $C(X)$ there is a minimal boundary $B \subset X$ of this type. Minimality means that $B = Y$ whenever $Y$ is a boundary for $H$ such that $Y \subset B$.

Proposition. 2.1. For every subcone $H$ of $C(X)$ there exists a minimal boundary $B \subset X$.

Proof: Let $B$ denote the (non-empty) collection of all boundaries for $H$, ordered by set inclusion and let $\mathcal{C}$ be a downward chain in $B$. We shall verify that

$$C_0 = \bigcap \{C \in \mathcal{C}\}$$
is a lower bound for \( \mathcal{C} \) in \( \mathcal{B} \). Indeed, \( C_0 \) is closed in \( X \) and a subset of all sets in \( \mathcal{C} \). For a function \( f \in H \) let
\[
Y_f = \{ y \in X | f(y) = \max(f,X) \}.
\]
This is a non-empty compact subset of \( X \), and \( Y_f \cap B \neq \emptyset \) for every boundary \( B \in \mathcal{B} \). If we had \( Y_f \cap C_0 = \emptyset \), then we would have \( Y_f \cap C = \emptyset \) for some \( C \in \mathcal{C} \) by the finite intersection property of closed sets in a compact space. Thus \( Y_f \cap C \neq \emptyset \) and
\[
\max(f,C_0) = \max(f,X).
\]
Thus \( C_0 \in \mathcal{C} \) as claimed. Following Zorn’s Lemma, \( \mathcal{B} \) then contains a minimal element. □

A minimal boundary of a subcone is, however, not necessarily unique, as the following example will show.

**Example 2.2.** Let \( X = [-1, +1] \) and let \( H \) be the subspace of all even functions in \( C([-1, +1]) \). That is
\[
H = \{ f \in C(X) | f(x) = f(-x) \text{ for all } x \in X \}.
\]
Then \( B = [0,1] \) is a minimal boundary for \( H \). Indeed, every function \( f \in H \) obviously takes its maximum (and minimum) value on \( B \). On the other hand, if \( Y \) is a closed subset of \( B \) such that \( Y \neq B \), then the open complement \( Y^c \) of \( Y \) contains a point \( 0 \leq x \in B \) and its negative \( -x \), and there is \( \varepsilon > 0 \) such that both intervals \( (x - \varepsilon, x + \varepsilon) \) and \( (-x - \varepsilon, -x + \varepsilon) \) are contained in \( Y^c \). There is \( f \in C([-1, +1]) \) such that \( f(x) = 1 \) and \( f(y) = 0 \) for all \( y \notin (x - \varepsilon, x + \varepsilon) \). The function
\[
y \to f(y) + f(-y)
\]
is in \( H \) and attains its maximum value outside \( Y \). Thus \( Y \) is not a boundary for \( H \). A similar argument shows that \( B' = [-1,0] \) is also a minimal boundary for \( H \), and these boundaries are therefore not unique in this case.

This deficit can however be remedied if we impose an additional assumption on the subcone \( H \) of \( C(X) \). We shall say that \( H \) (symmetrically) separates the points of \( X \) if for any two distinct points \( x, y \in X \) there is a function \( f \in H \) such that \( f(x) < f(y) \). Note that for a vector subspace \( H \) this notion coincides with the usual one, that is: for any two distinct points \( x, y \in X \) there is a function \( f \in H \) such that \( f(x) \neq f(y) \).

**Lemma 2.3.** Let \( H \) be a subcone of \( C(X) \) which separates the points of \( X \).

(a) For any two distinct points \( x, y \in X \) and \( \alpha \in \mathbb{R} \) there is a function \( f \in H \) such that \( f(y) = f(x) + \alpha \).

(b) For a compact subset \( K \) of \( X \) and \( x \in X \setminus K \) there are functions \( f_1, ..., f_n \in H \) such that the open neighborhood of \( x \)
\[
U = \{ y \in X | f_i(y) < f_i(x) + 1 \text{ for } i = 1, ..., n \}
\]
is disjoint from \( K \).

**Proof:** (a) Let \( x \) and \( y \) be distinct points of \( X \) and \( \alpha \in \mathbb{R} \). Since \( H \) separates the points of \( X \) we can choose a function \( h \in H \) such that either \( h(x) < h(y) \), in the case that \( \alpha \geq 0 \), or \( h(x) > h(y) \), in the case that \( \alpha < 0 \). The function
\[
f = \frac{\alpha}{h(y) - h(x)} h \in H
\]
has the required property.

(b) Let \( K \) be a compact subset of \( X \) and \( x \in X \setminus K \). For every \( y \in K \) there is by Part (a) a function \( f_y(y) = f_y(x) + 2 \). Set
\[
U_y = \{ z \in X | f_y(z) > f_y(x) + 1 \}
\]
The family \( \{U_y\}_{y \in K} \) forms an open cover for \( K \) and therefore contains a finite subcover \( U_1, ..., U_n \) corresponding to the functions \( f_1, ..., f_n \in H \). These functions satisfy the claim of Part (b). Indeed, the open set
\[
U = \{ y \in X | f_i(y) < f_i(x) + 1 \text{ for } i = 1, ..., n \}
\]
contains the point \( x \) and is disjoint from \( K \), since for every \( y \in K \) at least one of the functions \( f_i \) has the property that \( f_i(y) > f_i(x) + 1 \). □

**Proposition 2.4.** For a subcone \( H \) of \( C(X) \) which separates the points of \( X \) there exists a unique
minimal boundary \( B \), that is every other boundary for \( H \) contains \( B \).

**Proof:** We have to verify only uniqueness. Let \( B \) be a minimal boundary for \( H \) and let \( Y \) be a second boundary. Let us assume to the contrary of our claim that \( B \not\subseteq Y \). Then there is \( x_0 \in B \setminus Y \). Following Lemma 3 (b) there are \( f_1, ..., f_n \in H \) such that

\[
U = \{ y \in X \mid f_i(y) < f_i(x_0) + 1 \text{ for } i = 1, ..., n \}
\]

contains \( x_0 \) and is disjoint from \( Y \). The set

\[
B \setminus U = B \cap (X \setminus U)
\]

is closed and is a proper subset of \( B \), since it does not contain \( x_0 \in B \). Therefore due to the minimality of \( B \) it is not a boundary for \( H \). Thus we can find a function \( f \in H \) such that

\[
\max(f, B \setminus U) < \max(f, X).
\]

On the other hand since \( Y \) is a boundary for \( H \) we can find \( y \in Y \) such that

\[
f(y) = \max(f, X),
\]

and since \( y \not\in U \) there is \( k \in \{1, ..., n\} \) such that \( f_k(y) \geq f_k(x_0) + 1 \). Next we choose \( \alpha \geq 0 \) and consider the function \( g = \alpha f + f_k \in H \).

If \( x \in U \) then

\[
\alpha f(x) + f_k(x) < \alpha \max(f, X) + f_k(x_0) + 1.
\]

If \( x \in B \setminus U \), then

\[
\alpha f(x) + f_k(x) \leq \alpha \max(f, B \setminus U) + \max(f_k, X).
\]

Thus if we choose \( \alpha \geq 0 \) such that

\[
\alpha(\max(f, X) - \max(f, B \setminus U)) > \max(f_k, X) - f_k(x_0) - 1
\]

then we have

\[
\alpha f(x) + f_k(x) < \alpha \max(f, X) + f_k(x_0) + 1
\]

for all \( x \in B \), and hence

\[
\max(\alpha f + f_k, X) = \max(\alpha f + f_k, B) < \alpha \max(f, X) + f_k(x_0) + 1.
\]

since \( B \) is a boundary for \( H \). On the other hand we have

\[
\alpha f(y) + f_k(y) = \alpha \max(f, X) + f_k(y) \geq \alpha \max(f, X) + f_k(x_0) + 1
\]

Thus

\[
\max(\alpha f + f_k, X) \geq \alpha \max(f, X) + f_k(x_0) + 1,
\]

contradicting the above. \( \square \)

The unique minimal boundary of a subcone of \( C(X) \), if it exists, is also called the Šilov boundary of this subcone.

Integral representations for linear functionals

A *linear functional* \( I \) on a subcone \( H \) of \( C(X) \) is a mapping \( I : H \rightarrow \mathbb{R} \) such that

\[
I(f + g) = I(f) + I(g) \quad \text{and} \quad I(\alpha f) = \alpha I(f)
\]

for all \( f, g \in H \) and \( \alpha \geq 0 \). A linear functional \( I \) on \( H \) is called *u-continuous* if there is a constant \( C \geq 0 \) such that

\[
I(f) \leq I(g) + C \quad \text{whenever} \quad f \leq g + 1
\]

for \( f, g \in H \). This condition implies that \( I \) is *monotone*, that is

\[
I(f) \leq I(g) \quad \text{whenever} \quad f \leq g
\]

for \( f, g \in H \). We observe the following:

**Lemma. 3.1.** If the subcone \( H \) of \( C(X) \) contains a strictly positive function \( f_0 \), then every monotone linear functional on \( H \) is continuous.

**Proof:** Let \( I \) be a monotone linear functional on \( H \) and let \( f_0 \in H \) be strictly positive. Thus

\[
\alpha = \min\{f_0(x) \mid x \in X\} > 0.
\]

Let \( f, g \in H \) such that \( f \leq g + 1 \). Then

\[
f \leq g + 1 \leq g + \frac{1}{\alpha} f_0,
\]

and therefore

\[
I(f) \leq I(g) + \frac{1}{\alpha} I(f_0)
\]

using the monotonicity of \( I \). \( \square \)

We shall use the classical Riesz-Markov representation theorem (see for example Theorem
II.1.2 in [3]) for linear functionals on $C(X)$ spaces in order to derive a more general result for linear functionals on a subcone $H$ of $C(X)$. The resulting representation measures are supported by a boundary for $H$.

**Theorem.** 3.2. Let $H$ be a subcone of $C(X)$ and let $B \subseteq X$ be a boundary for $H$. For every u-continuous linear functional $I$ on $H$ there exists a positive regular Borel measure $\mu$ on $X$ which is supported by $B$ and such that

\[ I(f) \leq \int_X f \, d\mu \quad \text{for all } f \in H. \]

**Proof:** Let $I$ be a u-continuous linear functional on $H$ and let $C \geq 0$ such that

\[ I(f) \leq I(g) + C \quad \text{whenever } f \leq g + 1 \]

for $f, g \in H$. For a function $f \in C(X)$ we denote by $f|_B$ its restriction to the subset $B$ of $X$. We have $\max(f|_B, B) = \max(f, X)$ for all $f \in H$, since $B$ is a boundary for $H$. We define a $\mathbb{R}$-valued sublinear functional $p$ on $C(B)$ by

\[ p(f) = C \max(f, B) \]

for all $f \in C(B)$ and a $(\mathbb{R} \cup -\infty)$-valued superlinear functional $q$ by

\[ q(f) = \sup \{ I(h) | h \in H, h|_B \leq f \} \]

for $f \in C(B)$. As usual, we set $\sup \emptyset = -\infty$. Moreover, $q$ does not take the value $+\infty$, since $h|_B \leq f$ for $f \in C(B)$ and $h \in H$ implies that $h|_B \leq \max(f, B)$, hence $h \leq \max(f, B)$ and therefore

\[ I(h) \leq C \max(f, B) = p(f), \]

using the u-continuity of $I$. This shows that

\[ q(f) \leq p(f) \quad \text{for all } f \in C(X). \]

The sublinearity of $p$ and the superlinearity of $q$ are easily checked. Let us verify just one of the requirements for $q$: If

\[ h_1|_B \leq f \quad \text{and} \quad h_2|_B \leq g \]

for $h_1, h_2 \in H$ and $f, g \in C(B)$, then

\[ h_1|_B + h_2|_B \leq f + g, \]

hence

\[ I(h_1) + I(h_2) \leq q(f + g) \]

and therefore

\[ q(f) + q(g) \leq q(f + g). \]

Now according to the sandwich version of the Hahn-Banach theorem (see for example Corollary 1.3.26 in [3]) there exists a linear functional $L$ on $C(B)$ such that

\[ q(f) \leq L(f) \leq p(f) \]

for all $f \in C(X)$. We observe the following:

(i) $L$ is bounded, that is continuous. Indeed, if $f \leq 1$ for $f \in C(B)$, then $L(f) \leq p(f) \leq C$, hence if $\|f\| \leq 1$ then $|L(f)| \leq C$.

(ii) $L$ is monotone. Indeed, if $f \leq 0$ for $f \in C(B)$ then $L(f) \leq p(f) \leq 0$, hence if $f \leq g$ for $f, g \in C(B)$ then $f - g \leq 0$, and therefore

\[ L(f) - L(g) = L(f - g) \leq 0. \]

(iii) $L(f|_B) \geq I(f)$ for all $f \in H$. Indeed, following the definition of the superlinear functional $q$ we have

\[ L(f|_B) \geq q(f|_B) \geq I(f). \]

Next we apply the Riesz-Markov representation theorem (see Theorem II.1.2 in [3]): there is a regular Borel measure $\tilde{\mu}$ on $B$ such that

\[ L(f) = \int_B f \, d\tilde{\mu} \quad \text{for all } f \in C(B). \]

The measure $\tilde{\mu}$ on $B$ corresponds to a regular Borel measure $\mu$ on $X$ if we set

\[ \mu(A) = \tilde{\mu}(B \cap A) \]

for every Borel subset $A$ of $X$. This yields

\[ \int_X f \, d\mu = \int_B f \, d\tilde{\mu} = \int_B f|_B \, d\tilde{\mu} \]

for all $f \in C(X)$, and in particular
\[ \int_X f \, d\mu = \int_B f \, d\mu = \int_B f \, d\mu = L(f|_B) \geq I(f) \]

for all \( f \in H \), our claim. \( \Box \)

The statement of Theorem 3.2 can be further developed in the case that \( H \) is indeed a vector subspace of \( \mathcal{C}(X) \). It has been shown (see Theorem 3.3 and Corollary 4.4 in [5]) that in this case every continuous linear functional \( I \) on the subspace \( H \) of \( \mathcal{C}(X) \) can be expressed as a difference of two u-continuous ones, that is there are u-continuous linear functionals \( I_1 \) and \( I_2 \) on \( H \) such that

\[ I(f) = I_1(f) - I_2(f) \]

for all \( f \in H \). Using Theorem 3.2 the functionals \( I_1 \) and \( I_2 \) can be represented by positive regular Borel measures \( \mu_1 \) and \( \mu_2 \), respectively. That is, we have

\[ I_1(f) = \int_X f \, d\mu_1 \quad \text{and} \quad I_2(f) = \int_X f \, d\mu_2 \]

for all \( f \in H \). Equality in these representations follows since \( -f \in H \) whenever \( f \in H \). Consequently, the signed measure \( \mu = \mu_1 - \mu_2 \) is supported by \( B \) and represents the functional \( I \) on \( H \), that is

\[ I(f) = I_1(f) - I_2(f) = \int_X f \, d\mu_1 - \int_X f \, d\mu_2 \]

\[ = \int_X f \, d\mu \]

for all \( f \in H \). We summarize:

**Corollary.** 3.3. Let \( H \) be a linear subspace of \( \mathcal{C}(X) \) and let \( B \subset X \) be a boundary for \( H \). For every bounded linear functional \( I \in H^* \) there exists a regular Borel measure \( \mu \) on \( X \) which is supported by \( B \) and such that

\[ I(f) = \int_X f \, d\mu \quad \text{for all} \quad f \in H. \]

**Examples 3.4.** (a) Let \( X \) be the closed unit disc in \( \mathbb{R}^2 \) and let \( H \) be the subcone of \( \mathcal{C}(X) \) consisting of those functions \( f \in \mathcal{C}(X) \) that are subharmonic in the interior of \( X \), that is

\[ \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) \geq 0 \]

for all \((x, y)\) in the interior of \( X \). The subcone \( H \) symmetrically separates the points of \( X \) and contains the constants, and it is well known that its minimal boundary (Šilov boundary) consists of the circle line in this case, that is

\[ B = \{(x, y) \mid x^2 + y^2 = 1\}. \]

According to Theorem 3.2 every monotone (therefore u-continuous by Lemma 3.1) linear functional on \( H \) can be represented by a positive regular Borel measure on \( B \). This is best dealt with in polar coordinates \((r, \phi)\), where the subharmonic inequality translates into

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} \geq 0 \]

For a point evaluation at a point in the interior of \( X \) with the polar coordinates \((r, \theta)\), that is \( r < 1 \), this representation is given by the Poisson Integral Formula

\[ f(r, \theta) \leq \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{f(1, \phi)}{1 - 2r \cos(\theta - \phi) + r^2} \, d\phi \]

for every subharmonic function \( f \in H \). The representation measure \( \mu \) on \( B \) for this point evaluation is therefore the Lebesgue measure with the density function

\[ (1, \phi) \rightarrow \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2}. \]

For the subspace \( L \) of all harmonic functions, that is \( L = H \cap (-H) \), the above inequality turns into an equality, that is we have
\[
f(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} f(1, \phi) \frac{1}{1 - 2r \cos(\theta - \phi) + r^2} \, d\phi
\]

for every harmonic function \( f \in L \).

(b) Let \( X \) be a compact convex subset of a normed space (or a Hausdorff locally convex topological vector space). Recall that convexity means that \( \lambda x + (1 - \lambda)y \in X \) whenever \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \). An extreme point of \( X \) is a point \( x \in X \) such that

\[
x = \lambda y + (1 - \lambda)z
\]

for \( y, z \in X \) and \( 0 < \lambda < 1 \) implies that \( x = y = z \), that is \( x \) is not an interior point of a line segment in \( X \). A function \( f : X \to \mathbb{R} \) is said to be convex if

\[
f(x) \leq \lambda f(y) + (1 - \lambda)f(z)
\]

whenever \( x = \lambda y + (1 - \lambda)z \) for \( y, z \in X \) and \( 0 \leq \lambda \leq 1 \). The subcone \( H \) of all convex functions in \( C(X) \) symmetrically separates the points of \( X \) (this follows from the Hahn-Banach theorem) and contains the constants. According to the Krein-Milman theorem its minimal boundary \( B \) is the closure of the set of all extreme points of \( X \).

For a concrete example let \( X \) be a closed convex polygon in \( \mathbb{R}^2 \) with the vertices \( P_1, ..., P_n \). Then \( B = \{P_1, ..., P_n\} \) is the Šilov boundary for \( H \) and according to Theorem 3.2 every monotone linear functional \( I \) on \( H \) can be represented by a regular Borel measure \( \mu \) on \( B \). But the measures on the finite set \( B \) are just linear combinations of point evaluations \( \delta_{P_i} \). If in particular the functional \( I \) is monotone, and therefore u-continuous, then \( \mu \) is a convex combination of these point evaluations, that is

\[
\mu = \lambda_1 \delta_{P_1} + \cdots + \lambda_n \delta_{P_n},
\]

where \( \lambda_1, ..., \lambda_n \geq 0 \) and \( \lambda_1 + \cdots + \lambda_n = I(1) \).

Thus

\[
I(f) \leq \int_X f \, d\mu = \lambda_1 f(P_1) + \cdots + \lambda_n f(P_n)
\]

for all \( f \in H \). For affine functions, that is functions in \( L = H \cap (-H) \), and a continuous (not necessarily monotone) linear functional \( I \) on \( L \) we obtain according to Corollary 3.3 a similar representation, that is

\[
I(f) = \int_X f \, d\mu = \lambda_1 f(P_1) + \cdots + \lambda_n f(P_n)
\]

where \( P_i \in B \) and \( \lambda_i \in \mathbb{R} \) for \( i = 1, ..., n \).

References


