Maximum boundaries for cones of continuous functions on a compact space and integral representations for linear functionals

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Abstract

We present a simplified and easily accessible approach to the integral representation for continuous linear functionals on a cone of continuous real-valued functions on a compact set. The measures defining these integrals are supported by the maximum boundary of the respective cones.

Index Terms: spaces and cones of continuous functions, integral representation

1. Introduction

The concept of a maximum boundary for an algebra of continuous functions on a compact space was first proposed by Georgii Šilov in 1964 [6]. It was later generalized to vector spaces of continuous functions not necessarily closed for multiplication using rather demanding and complicated techniques from Choquet theory (see [1], [4] and [2]). These also generate our results concerning integral representations for continuous linear functionals on these spaces. We offer a much simplified and more easily accessible approach in this paper while also generalizing the concepts from linear spaces to cones of continuous functions.

2. Maximum Boundaries

Let X be a compact Hausdorff space and $C(X)$ the Banach space of all continuous functions on X endowed with the maximum norm, that is

$$\|f\| = \max\{|f(x)| \mid x \in X\}.$$

for $f \in C(X)$. A non-empty subset H of $C(X)$ is a called a *subcone* of $C(X)$ if

$$f + g \in H \quad \text{and} \quad \alpha f \in H,$$

whenever $f, g \in H$, and $\alpha \geq 0$. Linear subspaces are of course subcones in this sense. For a function $f \in C(X)$ and a closed subset Y of X we abbreviate

$$\max(f, Y) = \max\{|f(x)| \mid x \in Y\}.$$

Given a subcone H of $C(X)$, a closed subset Y of X is called a *(maximum) boundary for* H if

$$\max(f, Y) = \max(f, X)$$

holds for all $f \in H$, that is if all functions in H attain their maximum value on Y. If H is indeed a linear subspace of $C(X)$, then the functions in H also take their minimum values on Y, since a function $f \in H$ takes its minimum value where $-f \in H$ takes its maximum value. We shall use Zorn's Lemma to prove that for every subcone of $C(X)$ there is a minimal boundary $B \subset X$ of this type. Minimality means that $B = Y$ whenever Y is a boundary for H such that $Y \subset B$.

Proposition. 2.1. *For every subcone H of $C(X)$ there exists a minimal boundary $B \subset X$.*

Proof: Let $\mathcal{B}$ denote the (non-empty) collection of all boundaries for H, ordered by set inclusion and let $\mathfrak{C}$ be a downward chain in $\mathcal{B}$. We shall verify that

$$C_0 = \bigcap\{C \in \mathfrak{C}\}$$

is a lower bound for $\mathfrak{C}$ in $\mathcal{B}$. Indeed, C_0 is closed in X and a subset of all sets in $\mathfrak{C}$. For a function $f \in H$ let

$$Y_f = \{y \in X | \, f(y) = \max(f, X)\}.$$

This is a non-empty compact subset of X, and $Y_f \cap B \neq \emptyset$ for every boundary $B \in \mathcal{B}$. If we had $Y_f \cap C_0 = \emptyset$, then we would have $Y_f \cap C = \emptyset$ for some $C \in \mathfrak{C}$ by the finite intersection property of closed sets in a compact space. Thus $Y_f \cap C \neq \emptyset$ and

$$\max(f, C_0) = \max(f, X).$$

Thus $C_0 \in \mathfrak{C}$ as claimed. Following Zorn's Lemma, $\mathcal{B}$ then contains a minimal element. □

A minimal boundary of a subcone is, however, not necessarily unique, as the following example will show.

Example 2.2. Let $X = [-1, +1]$ and let H be the subspace of all even functions in $C([-1, +1])$, that is

$$H = \{f \in C(X) | \, f(x) = f(-x) \text{ for all } x \in X\}.$$

Then $B = [0,1]$ is a minimal boundary for H. Indeed, every function $f \in H$ obviously takes its maximum (and minimum) value on B. On the other hand, if Y is a closed subset of B such that $Y \neq B$, then the open complement Y^c of Y contains a point $0 \leq x \in B$ and its negative $-x$, and there is $\varepsilon > 0$ such that both intervals $(x - \varepsilon, x + \varepsilon)$ and $(-x - \varepsilon, -x + \varepsilon)$ are contained in Y^c. There is $f \in C([-1, +1])$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \notin (x - \varepsilon, x + \varepsilon)$. The function

$$y \to f(y) + f(-y)$$

is in H and attains its maximum value outside Y. Thus Y is not a boundary for H. A similar argument shows that $B' = [-1,0]$ is also a minimal boundary for H, and these boundaries are therefore not unique in this case.

This deficit can however be remedied if we impose an additional assumption on the subcone H of $C(X)$. We shall say that H *(symmetrically) separates the points of* X if for any two distinct points $x, y \in X$ there is a function $f \in H$ such that $f(x) < f(y)$. Note that for a vector subspace H this notion coincides with the usual one, that is: for any two distinct points $x, y \in X$ there is a function $f \in H$ such that $f(x) \neq f(y)$.

Lemma. 2.3. *Let H be a subcone of $C(X)$ which separates the points of X.*

(a) For any two distinct points $x, y \in X$ and $\alpha \in \mathbb{R}$ there is a function $f \in H$ such that $f(y) = f(x) + \alpha$.

(b) For a compact subset K of X and $x \in X \backslash K$ there are functions $f_1, \ldots, f_n \in H$ such that the open neighborhood of x

$$U = \{y \in X | \, f_i(y) < f_i(x) + 1 \; for \; i = 1, \ldots, n\}$$

is disjoint from K.

Proof: (a) Let x and y be distinct points of X and $\alpha \in \mathbb{R}$. Since H separates the points of X we can choose a function $h \in H$ such that either $h(x) < h(y)$, in the case that $\alpha \geq 0$, or $h(x) > h(y)$, in the case that $\alpha < 0$. The function

$$f = \frac{\alpha}{h(y) - h(x)} h \in H$$

has the required property.

(b) Let K be a compact subset of X and $x \in X \backslash K$. For every $y \in K$ there is by Part (a) a function $f_y(y) = f_y(x) + 2$. Set

$$U_y = \{z \in X | \, f_y(z) > f_y(x) + 1\}$$

The family $(U_y)_{y \in K}$ forms an open cover for K and therefore contains a finite subcover $U_1, \ldots, U_n$ corresponding to the functions $f_1, \ldots, f_n \in H$. These functions satisfy the claim of Part (b). Indeed, the open set

$$U = \{y \in X | \, f_i(y) < f_i(x) + 1 \text{ for } i = 1, \ldots, n\}$$

contains the point x and is disjoint from K, since for every $y \in K$ at least one of the functions f_i has the property that $f_i(y) > f_i(x) + 1$. □

Proposition. 2.4. *For a subcone H of $C(X)$ which separates the points of X there exists a unique* *minimal boundary B, that is every other boundary for H contains B.*

Proof: We have to verify only uniqueness. Let B be a minimal boundary for H and let Y be a second boundary. Let us assume to the contrary of our claim that $B \not\subset Y$. Then there is $x_0 \in B \setminus Y$. Following Lemma 3 (b) there are $f_1, \dots, f_n \in H$ such that

$$U = \{y \in X |\ f_i(y) < f_i(x_0) + 1 \text{ for } i = 1, \dots, n\}$$

contains x_0 and is disjoint from Y. The set

$$B \setminus U = B \cap (X \setminus U)$$

is closed and is a proper subset of B, since it does not contain $x_0 \in B$. Therefore due to the minimality of B it is not a boundary for H. Thus we can find a function $f \in H$ such that

$$\max(f, B \setminus U) < \max(f, X).$$

On the other hand since Y is a boundary for H we can find $y \in Y$ such that

$$f(y) = \max(f, X),$$

and since $y \notin U$ there is $k \in \{1, \dots, n\}$ such that $f_k(y) \geq f_k(x_0) + 1$. Next we choose $\alpha \geq 0$ and consider the function $g = \alpha f + f_k \in H$.

If $x \in U$ then

$$\alpha f(x) + f_k(x) < \alpha \max(f, X) + f_k(x_0) + 1.$$

If $x \in B \setminus U$, then

$$\alpha f(x) + f_k(x) \leq \alpha \max(f, B \setminus U) + \max(f_k, X).$$

Thus if we choose $\alpha \geq 0$ such that

$$\alpha(\max(f, X) - \max(f, B \setminus U)) > \max(f_k, X) - f_k(x_0) - 1$$

then we have

$$\alpha f(x) + f_k(x) < \alpha \max(f, X) + f_k(x_0) + 1$$

for all $x \in B$, and hence

$$\max(\alpha f + f_k, X) = \max(\alpha f + f_k, B) < \alpha \max(f, X) + f_k(x_0) + 1,$$

since B is a boundary for H. On the other hand we have

$$\alpha f(y) + f_k(y) = \alpha \max(f, X) + f_k(y) \geq \alpha \max(f, X) + f_k(x_0) + 1$$

Thus

$$\max(\alpha f + f_k, X) \geq \alpha \max(f, X) + f_k(x_0) + 1,$$

contradicting the above. □

The unique minimal boundary of a subcone of $C(X)$, if it exists, is also called the *Šilov boundary* of this subcone.

Integral representations for linear functionals

A *linear functional* I on a subcone H of $C(X)$ is a mapping $I : H \to \mathbb{R}$ such that

$$I(f + g) = I(f) + I(g) \quad \text{and} \quad I(\alpha f) = \alpha I(f)$$

for all $f, g \in H$ and $\alpha \geq 0$. A linear functional I on H is called *u-continuous* if there is a constant $C \geq 0$ such that

$$I(f) \leq I(g) + C \quad \text{whenever} \quad f \leq g + 1$$

for $f, g \in H$. This condition implies that I is *monotone*, that is

$$I(f) \leq I(g) \quad \text{whenever} \quad f \leq g$$

for $f, g \in H$. We observe the following:

Lemma. 3.1. If the subcone H of $C(X)$ contains a strictly positive function f_0, then every monotone linear functional on H is continuous.

Proof: Let I be a monotone linear functional on H and let $f_0 \in H$ be strictly positive. Thus

$$\alpha = \min\{f_0(x) |\ x \in X\} > 0.$$

Let $f, g \in H$ such that $f \leq g + 1$. Then

$$f \leq g + 1 \leq g + \frac{1}{\alpha} f_0,$$

and therefore

$$I(f) \leq I(g) + \frac{1}{\alpha} I(f_0)$$

using the monotonicity of I. □

We shall use the classical Riesz-Markov representation theorem (see for example Theorem

II.1.2 in [3]) for linear functionals on $C(X)$ spaces in order to derive a more general result for linear functionals on a subcone H of $C(X)$. The resulting representation measures are supported by a boundary for H.

Theorem. 3.2. *Let H be a subcone of $C(X)$ and let $B \subset X$ be a boundary for H. For every u-continuous linear functional I on H there exists a positive regular Borel measure μ on X which is supported by B and such that*

$$I(f) \le \int_X f\, d\mu \quad for\ all \quad f \in H.$$

Proof: Let I be a u-continuous linear functional on H and let $C \ge 0$ such that

$$I(f) \le I(g) + C \quad \text{whenever} \quad f \le g + 1$$

for $f, g \in H$. For a function $f \in C(X)$ we denote by $f|_B$ its restriction to the subset B of X. We have

$$\max(f|_B, B) = \max(f, X)$$

for all $f \in H$, since B is a boundary for H. We define a $\mathbb{R}$-valued sublinear functional p on $C(B)$ by

$$p(f) = C \max(f, B)$$

for all $f \in C(B)$ and a $(\mathbb{R} \cup -\infty)$-valued superlinear functional q by

$$q(f) = \sup\{I(h) |\ h \in H, h|_B \le f\}$$

for $f \in C(B)$. As usual, we set $\sup\emptyset = -\infty$. Moreover, q does not take the value $+\infty$, since $h|_B \le f$ for $f \in C(B)$ and $h \in H$ implies that $h|_B \le \max(f, B)$, hence $h \le \max(f, B)$ and therefore

$$I(h) \le C \max(f, B) = p(f),$$

using the u-continuity of I. This shows that

$$q(f) \le p(f) \quad \text{for all } f \in C(X).$$

The sublinearity of p and the superlinearity of q are easily checked. Let us verify just one of the requirements for q: If

$$h_1|_B \le f \quad \text{and} \quad h_2|_B \le g$$

for $h_1, h_2 \in H$ and $f, g \in C(B)$, then

$$h_1|_B + h_2|_B \le f + g,$$

hence

$$I(h_1) + I(h_2) \le q(f + g)$$

and therefore

$$q(f) + q(g) \le q(f + g).$$

Now according to the sandwich version of the Hahn-Banach theorem (see for example Corollary I.3.26 in [3]) there exists a linear functional L on $C(B)$ such that

$$q(f) \le L(f) \le p(f)$$

for all $f \in C(X)$. We observe the following:

(i) L is bounded, that is continuous. Indeed, if $f \le 1$ for $f \in C(B)$, then $L(f) \le p(f) \le C$, hence if $\|f\| \le 1$ then $|L(f)| \le C$.

(ii) L is monotone. Indeed, if $f \le 0$ for $f \in C(B)$ then $L(f) \le p(f) \le 0$, hence if $f \le g$ for $f, g \in C(B)$ then $f - g \le 0$, and therefore

$$L(f) - I(g) = L(f - g) \le 0.$$

(iii) $L(f|_B) \ge I(f)$ for all $f \in H$. Indeed, following the definition of the superlinear functional q we have

$$L(f|_B) \ge q(f|_B) \ge I(f).$$

Next we apply the Riesz-Markov representation theorem (see Theorem II.1.2 in [3]): there is a regular Borel measure $\tilde{\mu}$ on B such that

$$L(f) = \int_B f\, d\tilde{\mu} \quad \text{for all} \quad f \in C(B).$$

The measure $\tilde{\mu}$ on B corresponds to a regular Borel measure μ on X if we set

$$\mu(A) = \tilde{\mu}(B \cap A)$$

for every Borel subset A of X. This yields

$$\int_X f\, d\mu = \int_B f\, d\mu = \int_B f|_B\, d\tilde{\mu}$$

for all $f \in C(X)$, and in particular

$$\int_X f\,d\mu = \int_B f\,d\mu = \int_B f|_B\,d\tilde{\mu} = L(f|_B) \geq I(f)$$

for all $f \in H$, our claim. □

The statement of Theorem 3.2 can be further developed in the case that H is indeed a vector subspace of $C(X)$. It has been shown (see Theorem 3.3 and Corollary 4.4 in [5]) that in this case every continuous linear functional I on the subspace H of $C(X)$ can be expressed as a difference of two u-continuous ones, that is there are u-continuous linear functionals I_1 and I_2 on H such that

$$I(f) = I_1(f) - I_2(f)$$

for all $f \in H$. Using Theorem 3.2 the functionals I_1 and I_2 can be represented by positive regular Borel measures μ_1 and μ_2, respectively. That is, we have

$$I_1(f) = \int_X f\,d\mu_1 \quad \text{and} \quad I_2(f) = \int_X f\,d\mu_2$$

for all $f \in H$. Equality in these representations follows since $-f \in H$ whenever $f \in H$. Consequently, the signed measure $\mu = \mu_1 - \mu_2$ is supported by B and represents the functional I on H, that is

$$I(f) = I_1(f) - I_2(f) = \int_X f\,d\mu_1 - \int_X f\,d\mu_2 = \int_X f\,d\mu$$

for all $f \in H$. We summarize:

Corollary. 3.3. *Let H be a linear subspace of $C(X)$ and let $B \subset X$ be a boundary for H. For every bounded linear functional $I \in H^*$ there exists a regular Borel measure μ on X which is supported by B and such that*

$$I(f) = \int_X f\,d\mu \quad \textit{for all} \quad f \in H.$$

Examples 3.4. (a) Let X be the closed unit disc in $\mathbb{R}^2$ and let H be the subcone of $C(X)$ consisting of those functions $f \in C(X)$ that are subharmonic in the interior of X, that is

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) \geq 0$$

for all (x,y) in the interior of X. The subcone H symmetrically separates the points of X and contains the constants, and it is well known that its minimal boundary (Šilov boundary) consists of the circle line in this case, that is

$$B = \{(x,y)|\ x^2 + y^2 = 1\}.$$

According to Theorem 3.2 every monotone (therefore u-continuous by Lemma 3.1) linear functional on H can be represented by a positive regular Borel measure on B. This is best dealt with in polar coordinates (r,ϕ), where the subharmonic inequality translates into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} \geq 0$$

For a point evaluation at a point in the interior of X with the polar coordinates (r,θ), that is $r < 1$, this representation is given by the Poisson Integral Formula

$$f(r,\theta) \leq \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(1,\phi)}{1-2r\cos(\theta-\phi)+r^2}\,d\phi$$

for every subharmonic function $f \in H$. The representation measure μ on B for this point evaluation is therefore the Lebesgue measure with the density function

$$(1,\phi) \to \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}.$$

For the subspace L of all harmonic functions, that is $L = H \cap (-H)$, the above inequality turns into an equality, that is we have

$$f(r,\theta) = \frac{1-r^2}{2\pi}\int_0^{2\pi} \frac{f(1,\phi)}{1-2r\cos(\theta-\phi)+r^2}\, d\phi$$

for every harmonic function $f \in L$.

(b) Let X be a compact convex subset of a normed space (or a Hausdorff locally convex topological vector space). Recall that convexity means that $\lambda x + (1-\lambda)y \in X$ whenever $x, y \in X$ and $0 \leq \lambda \leq 1$. An *extreme point* of X is a point $x \in X$ such that

$$x = \lambda y + (1-\lambda)z$$

for $y, z \in X$ and $0 < \lambda < 1$ implies that $x = y = z$, that is x is not an interior point of a line segment in X. A function $f : X \to \mathbb{R}$ is said to be *convex* if

$$f(x) \leq \lambda f(y) + (1-\lambda) f(z)$$

whenever $x = \lambda y + (1-\lambda)z$ for $y, z \in X$ and $0 \leq \lambda \leq 1$. The subcone H of all convex functions in $C(X)$ symmetrically separates the points of X (this follows from the Hahn-Banach theorem) and contains the constants. According to the Krein-Milman theorem its minimal boundary B is the closure of the set of all extreme points of X.

For a concrete example let X be a closed convex polygon in $\mathbb{R}^2$ with the vertices $P_1, \ldots, P_n$. Then $B = \{P_1, \ldots, P_n\}$ is the Šilov boundary for H and according to Theorem 3.2 every monotone linear functional I on H can be represented by a regular Borel measure μ on B. But the measures on the finite set B are just linear combinations of point evaluations δ_{P_i}. If in particular the functional I is monotone, and therefore u-continuous, then μ is a convex combination of these point evaluations, that is

$$\mu = \lambda_1 \delta_{P_1} + \cdots + \lambda_n \delta_{P_n},$$

where $\lambda_1, \ldots, \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = I(1)$. Thus

$$I(f) \leq \int_X f\, d\mu = \lambda_1 f(P_1) + \cdots + \lambda_n f(P_n)$$

for all $f \in H$. For *affine* functions, that is functions in $L = H \cap (-H)$, and a continuous (not necessarily monotone) linear functional I on L we obtain according to Corollary 3.3 a similar representation, that is

$$I(f) = \int_X f\, d\mu = \lambda_1 f(P_1) + \cdots + \lambda_n f(P_n)$$

where $P_i \in B$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n$.

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